

Dissipation Approximation and Viscous Potential Flow

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August 2001

Levich [1962] computed the drag on a rising gas bubble by computing the viscous dissipation on the potential flow of an inviscid fluid outside a moving sphere. Lamb [1924] computed the rate of decay of a free wave on an inviscid fluid by evaluating the dissipation. These dissipation calculations are approximations to the full Navier-Stokes equations at high Reynolds numbers or other conditions in which potential flow of an inviscid fluid is believed to be close to real flows.

Viscous potential flow is an exact potential flow solution of the Navier-Stokes in which the viscosity enters into potential flow solutions through the viscous term in the normal stress balance Joseph & Liao [1994]. The relation of the dissipation approximation to viscous potential flow is discussed here.

Consider the case of a liquid and gas in which the gas is passive having no dynamic consequence on the liquid. Let V the volume of the liquid and A is its boundary. The Navier-Stokes equations for the liquid are

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p + \mu \operatorname{div} \mathbf{D}, \quad \operatorname{div} \mathbf{u} = 0 \quad (1)$$

The mechanical energy equation for (1) is

$$\frac{\rho}{2} \frac{d}{dt} \int_V (u)^2 v = \int_A \mathbf{u} \cdot (\mathbf{I} \cdot \mathbf{n}) dA - \int_V 2\mu \mathbf{D} : \mathbf{D} dV \quad (2)$$

where

$$\mathbf{T} = -p\mathbf{1} + 2\mu \mathbf{D}[\mathbf{u}] \quad (3)$$

is the stress and $\mathbf{D}[\mathbf{u}]$ is the rate of strain.

In the case of the gas bubble it is assumed that the bubble is in steady flow. If it were a solid then every point on the sphere would move with the same velocity

$$\mathbf{u} = \mathbf{e}_x U \quad \text{for } \mathbf{x} \text{ on } A. \quad (4)$$

Then

$$UD = 2\mu \int_V \mathbf{D} : \mathbf{D} dV \quad (5)$$

where

$$\mathbf{D} = \int_A \mathbf{e}_x \cdot \mathbf{T} \cdot \mathbf{n} da \quad (6)$$

Equation (5) was used by Levich but (4) does not hold on A in the case of gas bubble. If this

approximation is employed, then

$$DU = 2\mu \int_V \mathbf{D} : \mathbf{D} dV = 2\mu \int_V \frac{\partial^2 \phi}{\partial x_i \partial x_j} \frac{\partial^2 \phi}{\partial x_i \partial x_j} dV \quad (7)$$

where

$$\mathbf{u} = \nabla \phi, \quad \phi = -\frac{1}{2} U a^3 \frac{\cos \theta}{r^2} \quad (8)$$

and

$$D = 12\pi a \mu U, \quad C_D = 48/R \quad (9)$$

Analysis of the rising bubble based on viscous potential flow is the same as for inviscid potential flow except that the viscous contribution to the normal stress balance must be included. This viscous contribution would lead to a distortion of the spherical shape of the bubble which could be computed as a perturbation from the spherical shape in powers of γ^{-1} where γ is the surface tension. Moore [1958] applied the normal stress boundary condition to a passive spherical bubble and found using (8)

$$\text{Pr } r = -p + 2\mu \frac{\partial u}{\partial r} = p_i - 6 \left(\frac{\mu U}{a} \right) \cos \theta \quad (10)$$

where p_i is the pressure from the potential flow solution. Counting the tangential stress of the potential flow on the bubble surface as zero, he computed

$$D = 8\pi \mu U a, \quad C_D = 32/R \quad (11)$$

In a later paper, Moore [1962] carried out a boundary layer analysis at the surface of the bubble and found that

$$C_D = \frac{48}{R} \left\{ 1 - \frac{2.2}{\sqrt{R}} + \dots \right\} \quad (12)$$

The leading order agrees with the Levich formula.

This problem has been reviewed by Batchelor [1967] who has compared (9) and (12) with experimental data (see our Figure 1). We added $C_D = 32/R$ as $\text{---}\dots\text{---}$; it is in rather better agreement with the data than (9) or (12).

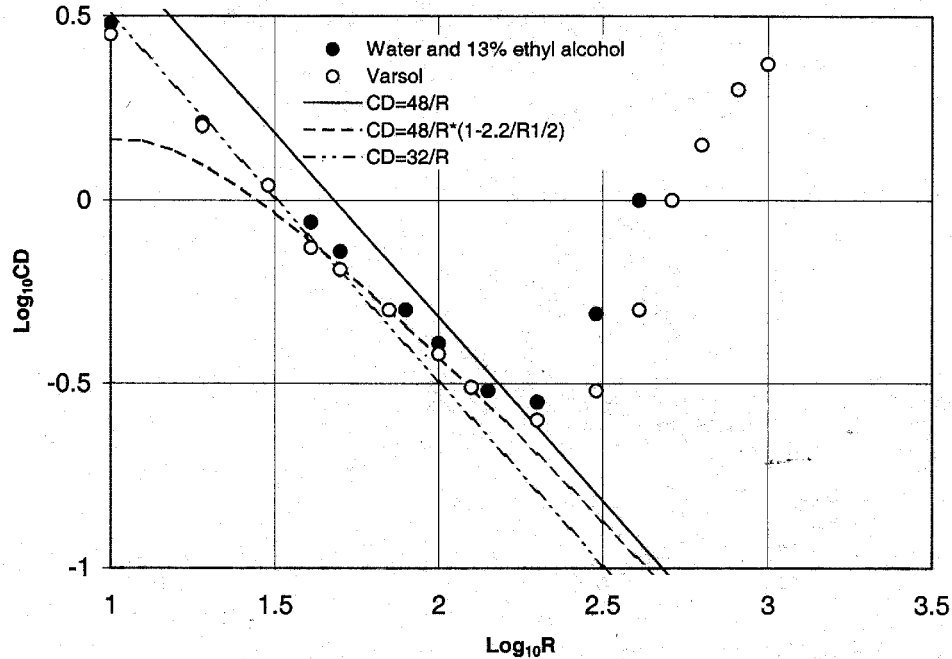


Figure 1. (after Batchelor 1967.) The drag coefficient of gas bubbles rising through liquids. The points for two particular liquids are taken from experimental curves given by Haberman and ??? (1953).

Lamb [1924, p. 624] considered the effect of the viscous dissipation of a free traveling wave given by the potential

$$\phi = ace^{ky} \cos k(x-ct) \quad (13)$$

and finds that the mean value of the dissipation per unit area is given

$$2\mu k^3 a^2 c^2 \quad (14)$$

“The kinetic energy per unit area is $\frac{1}{4} \rho k a^2 c^2$, and the total energy (kinetic plus potential) is therefore double of this. Hence in the absence of surface forces

$$\frac{d}{dt} \left(\frac{1}{2} \rho k c^2 a^2 \right) = -2\mu k^3 a^2 c^2, \quad (15)$$

$$\frac{da}{dt} = -2\nu k^2 a, \quad (16)$$

$$a = a_0 e^{-2\nu k^2 t} . ”$$

Equation (16) gives the rate of decay of a free wave on an inviscid fluid due to viscosity.

It is convenient to interpret Lamb’s results in terms of gravity waves for which

$$C = \sqrt{\frac{g}{k}} \quad (17)$$

When a gravity term $-\rho g$ is added to the right side of (1) the energy equation (2) may be written as

$$\frac{d}{dt}(\mathcal{E} + \mathcal{P}) = \int_0^{2\pi/k} \mathbf{u} \cdot \mathbf{\Gamma} \cdot \mathbf{n} - \int_{-\infty}^n \int_0^{2\pi/k} 2\mu \mathbf{D} : \mathbf{D} dy dx \quad (18)$$

where \mathcal{E} is the kinetic energy and \mathcal{P} the potential energy

$$\mathcal{P} = \int_0^{2\pi/k} \rho g \eta^2 dx \quad (19)$$

where $z - \eta(x, t) = 0$ (see Joseph 1976, p. 250). For the free motion of an inviscid potential the stress traction term will vanish and the left side of (18) can be computed in the linear case as in (15). The stress traction term could not be neglected for viscous potential flow.

An analysis of the stability of gravity waves using viscous potential is embedded in the analysis of Kelvin-Helmholtz instability by Funada & Joseph [2001]. A free wave is not stable, it must decay but at half the rate given by Lamb's dissipation calculation. In the analysis of linear stability of gravity waves based on viscous potential flow $\mathbf{u} = \nabla\phi$, $\nabla^2\phi = 0$ we find, after eliminating the pressure in the normal stress balance, that

$$\frac{\partial\phi}{\partial t} + g\eta + 2v\frac{\partial^2\phi}{\partial y^2} = 0 \quad (20)$$

and from the kinematic condition for $y = \eta$ we get

$$\frac{\partial\phi}{\partial y} = \frac{\partial\eta}{\partial t} \quad (21)$$

on $y = 0$. After eliminating η in (20) using (21) and applying normal modes (13) we find

$$C^2 + 2vick^2 - g/k = 0 \quad (22)$$

or

$$C = -ivk \pm \sqrt{\frac{g}{k} - v^2k^2} \quad (23)$$

Hence the normal solution is proportional to

$$e^{-vk^2t} e^{ik \left(x \pm t \sqrt{\frac{g}{k} - v^2k^2} \right)} \quad (24)$$

The amplitude of the wave decays at a rate

$$\frac{d\phi}{dt} = vk^2a, \quad (25)$$

one-half of the rate given by (16). The wave speed C is given by

$$C = \sqrt{\frac{g}{k} - v^2k^2}, \quad (26)$$

which is slower $\sqrt{g/k}$ for $k^3 < g/v^2$. For very large values of k , short standing waves do not

propagate but simply decay at a rate given

$$a = a_0 \exp\left\{-\frac{1}{2} \frac{g}{vk} t\right\}. \quad (27)$$

As far as I know there is no literature on the decay rate of gravity waves due to viscosity.

Acknowledgement

This work was supported by the Engineering Research Program of the Office of Basic Energy Sciences at the DOE, by the NSF/CTS-0076648, by an ARO grant DA/DAAH04 and by Intevp S.A.

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