Potential flow solutions of the compressible Navier-Stokes equations for sound waves

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Equations for sound waves are derived by linearizing the compressible Navier-Stokes equations on a state of rest. These linearized equations support a potential flow with the novel features that the Bernoulli equation and the potential as well as the stress depend on the viscosity. The effect of viscosity is to produce decay in time of spatially periodic waves or decay and growth in space of time periodic waves.

1. Potential flow solutions of the Navier-Stokes equations for viscous incompressible fluids

Potential flows $\mathbf{u} = \nabla\phi$ are solutions of the Navier-Stokes equations for viscous incompressible fluids. The viscous term $\mu\nabla^2\mathbf{u} = \mu\nabla\nabla^2\phi$ vanishes, but the viscous contribution to the stress in an incompressible fluid (Stokes 1850)

$$T_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = -p\delta_{ij} + 2\mu\frac{\partial^2\phi}{\partial x_i \partial x_j} \tag{1.1}$$

does not vanish in general. Not all models of viscoelastic fluids admit a potential flow solution; the curl of divergence of the extra stress must vanish. Potential flows of incompressible fluids admit a pressure (Bernoulli) equation when the divergence of the stress is a gradient as in inviscid fluids, viscous fluids, linear viscoelastic fluids and second order fluids (for which a term proportional to the square of the velocity gradient called a viscoelastic pressure appears). All of the classical results for inviscid potential flows hold for viscous potential flows with the caveat that the viscous stresses are not generally zero. The differences between inviscid and viscous and viscoelastic potential flow together with a review of the literature prior to 1994 are discussed by Joseph and Liao (1994a,b).

Potential flows will not generally satisfy boundary conditions which are associated with the requirement that the tangential component of velocity and the shear stress should be continuous across the interface separating the fluid from a solid or another fluid. The velocity and pressure are in viscous incompressible potential flow the same as in inviscid incompressible potential flow when fluid-fluid interfaces or free surfaces are not present.

The viscosity enters explicitly into the problem formulation for interface problems through the viscous term in the normal stress balance across the interface. Viscous potential flow analysis gives good approximations to fully viscous flows in cases where the shear from the gas flow is negligible; the Rayleigh-Plesset bubble is a potential flow which satisfies the Navier-Stokes equations and all the interface conditions. Joseph, Belanger and Beavers (1999) constructed a viscous potential flow of the Rayleigh-Taylor instability which is almost indistinguishable from the exact fully viscous analysis. Joseph, Beavers and Funada (2002) constructed a viscoelastic potential flow analysis for the Rayleigh-Taylor instability of an Oldroyd-B model fluid which is also in very good agreement with the unapproximated solution. The two papers just mentioned were applied to experiments on drop breakup at very high Weber numbers and gave rise to satisfying agreements.

Funada and Joseph (2001) gave a viscous potential flow analysis of Kelvin-Helmholtz instability in a channel. There is no exact solution for the linearized viscous equations for this problem but a number of approximate solutions have been given. Mata, Pereyra, Trallero and Joseph (2002) compared these theories with experiments. The theories do not agree with each other and only the viscous potential flow solution of Funada and Joseph agrees with the experiments.

Funada and Joseph (2002a) gave a viscous potential flow analysis of capillary instability. Results of linearized analysis based on potential flow of a viscous and inviscid fluid were compared with the unapproximated normal mode analysis of the linearized Navier-Stokes equations. The growth rates for the inviscid fluid are largest, the growth rates of the fully viscous problems are smallest and those of viscous potential flow are between. The growth rates of the fully viscous fluid analysis and viscous potential flow are uniformly in good agreement. The results from all three theories converge when a Reynolds number $\gamma D\rho_l/\mu_l^2$ based on the velocity γ/μ_l of capillary collapse is large, $(\gamma, D, \rho_l, \mu_l)$ = (surface tension, diameter, density, viscosity). The convergence results apply to two liquids as well as to liquid and gas. Funada and Joseph (2002b) did the same type of analysis of capillary instability of a viscoelastic fluid of the Maxwell model. The results are similar to those for viscous potential flow.

In a recent paper Joseph (2002) applied the theory of viscous potential flow to the problem of finding the rise velocity U of a spherical cap bubble (Davies and Taylor 1950, Batchelor 1967). The rise velocity is given by

$$\frac{U}{\sqrt{gD}} = -\frac{8}{3}\frac{\nu(1+8s)}{\sqrt{gD^3}} + \frac{\sqrt{2}}{3}\left[1 - 2s - \frac{16s\sigma}{\rho g D^2} + \frac{32\nu^2}{gD^3}(1+8s)^2\right]^{1/2} \tag{1.2}$$

where $R = D/2$ is the radius of the cap, ρ and ν are the density and kinematic viscosity of the liquid, σ is surface tension and $s = r''(0)/D$ is the deviation of the free surface

$$r(\theta) = R + \frac{1}{2}r''(0)\theta^2 = R(1 + s\theta^2) \tag{1.3}$$

from perfect sphericity $r(\theta) = R$ near the stagnation point $s = 0$. The bubble nose is more pointed when $s < 0$ and blunted when $s > 0$. A more pointed bubble increases the rise velocity; the blunter bubble rises slower. The Davies-Taylor (1950) result $U = \frac{\sqrt{2}}{3}\sqrt{gD}$ arises when all other effects vanish; if s alone is zero,

$$\frac{U}{\sqrt{gD}} = -\frac{8}{3}\frac{\nu}{\sqrt{gD^3}} + \frac{\sqrt{2}}{3}\left[1 + \frac{32\nu^2}{gD^3}\right]^{1/2} \tag{1.4}$$

showing that viscosity slows the rise velocity.

2. Potential flow solutions of the Navier-Stokes equations for viscous compressible fluids

Potential flows are not in general solutions of the compressible Navier-Stokes equations. To have such solutions it is necessary to show that curl $\mathbf{u} = 0$ is a solution (see Joseph and Liao 1994a) of the vorticity equation. The gradients of density and viscosity which are spoilers for the general vorticity equation do not enter into the equations which perturb the state of rest with uniform pressure p_0 and density ρ_0.

The stress for a compressible viscous fluid is given by

$$T_{ij} = -\left(p + \frac{2}{3}\mu \text{div}\mathbf{u}\right)\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2.1}$$

Here, the second coefficient of viscosity is selected so that $T_{ii} = -3p$. (The results to follow will apply also to the case when other choices are made for the second coefficient of viscosity.)

The equations of motion are given by

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = \text{div}\mathbf{T} \tag{2.2}$$

together with

$$\frac{\partial \rho}{\partial t} + \mathbf{u}\cdot\nabla\rho + \rho\text{div}\mathbf{u} = 0 \tag{2.3}$$

To study acoustic propagation, these equations are linearized; putting

$$[\mathbf{u}, p, \rho] = [\mathbf{u}', p_0 + p', \rho_0 + \rho'] \tag{2.4}$$

where $\mathbf{u}', p'$ and ρ' are small quantities, we get

$$T_{ij} = -\left(p_0 + p' + \frac{2}{3}\mu_0 \text{div}\mathbf{u}'\right)\delta_{ij} + \mu_0\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right) \tag{2.5}$$

$$\rho_0 \frac{\partial \mathbf{u}'}{\partial t} = -\nabla p' + \mu_0\left(\nabla^2 \mathbf{u}' + \frac{1}{3}\nabla \text{div}\mathbf{u}'\right) \tag{2.6}$$

$$\frac{\partial \rho'}{\partial t} + \rho_0 \text{div}\mathbf{u}' = 0 \tag{2.7}$$

where p_0, ρ_0 and μ_0 are constants. For acoustic problems, we assume that a small change in ρ induces small changes in p by fast adiabatic processes; hence

$$p' = C_0^2 \rho' \tag{2.8}$$

where C_0 is the speed of sound.

Forming now the curl of (2.6), we find that

$$\rho_0 \frac{\partial \zeta}{\partial t} = \mu_0 \nabla^2 \zeta, \quad \zeta = \text{curl}\mathbf{u}'. \tag{2.9}$$

Hence $\zeta = 0$, is a solution of the vorticity equation and we may introduce a potential

$$\mathbf{u}' = \nabla \phi. \tag{2.10}$$

Combining next (2.10) and (2.6), we get

$$\nabla\left[\rho_0 \frac{\partial \phi}{\partial t} + p' - \frac{4}{3}\mu_0 \nabla^2 \phi\right] = 0 \tag{2.11}$$

The quantity in the bracket is equal to an arbitrary function of the time which may be absorbed in ϕ.

A viscosity dependent Bernoulli equation

$$\rho_0 \frac{\partial \phi}{\partial t} + p' - \frac{4}{3}\mu_0 \nabla^2 \phi = 0 \tag{2.12}$$

is implied by (2.11). The stress (2.5) is given in terms of the potential ϕ by

$$T_{ij} = -\left(p_0 - \rho_0 \frac{\partial \phi}{\partial t} + 2\mu_0 \nabla^2 \phi\right)\delta_{ij} + 2\mu_0 \frac{\partial^2 \phi}{\partial x_i \partial x_j}. \tag{2.13}$$

To obtain the equation satisfied by the potential ϕ, we eliminate ρ' in (2.7) with p' using (2.8), then eliminate $\mathbf{u}' = \nabla\phi$ and p' in terms of ϕ using (2.12) to find

$$\frac{\partial^2 \phi}{\partial t^2} = \left(C_0^2 + \frac{4}{3}\nu_0 \frac{\partial}{\partial t} \right) \nabla^2 \phi \tag{2.14}$$

where the potential ϕ depends on the speed of sound and the kinematic viscosity $\nu_0 = \mu_0/\rho_0$.

The damped wave equation (2.14) may be derived directly without introducing a potential from the compressible Navier-Stokes equation in the acoustic approximation; obviously, the viscosity dependent Bernoulli equation (2.12) requires one to introduce a potential. Lamb (1932) derived (2.14) for the velocity in one space dimension directly from the compressible linearized Navier-Stokes equation (2.6) for plane waves in a laterally unbounded medium (his equation (4), page 647). Lighthill (1978) derived the same one-dimensional damped wave equation for the density rather than the velocity without introducing a velocity potential. In Lighthill's equation (205), $\frac{4}{3}\nu_0$ is replaced by δ, a relaxation time for a relaxing gas given by his equation (200), which may be written as

$$p' = C_0^2 \rho' + \delta \frac{\partial \rho'}{\partial t}. \tag{2.15}$$

Inserting (2.15) into (2.12) we get

$$\rho_0 \frac{\partial \phi}{\partial t} + C_0^2 \rho' + \delta \frac{\partial \rho'}{\partial t} - \frac{4}{3}\mu_0 \nabla^2 \phi = 0. \tag{2.16}$$

Combining now (2.16) with

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla^2 \phi = 0,$$

we find a generalized damped wave equation

$$\frac{\partial^2 \phi}{\partial t^2} = \left(C_0^2 + \left[\delta + \frac{4}{3}\nu_0 \right] \frac{\partial}{\partial t} \right) \nabla^2 \phi. \tag{2.17}$$

A dimensionless form for the potential equation (2.17)

$$\frac{\partial^2 \phi}{\partial T^2} = \left(1 + \frac{\partial}{\partial T}\right) \nabla^2 \phi, \qquad \nabla^2 \phi = \frac{\partial^2 \phi}{\partial X^2} + \frac{\partial^2 \phi}{\partial Y^2} + \frac{\partial^2 \phi}{\partial Z^2} \tag{2.18}$$

arises from a change of variables

$$t = \frac{\left[\delta + \frac{4}{3}\nu_0\right]}{C_0^2} T, \qquad \mathbf{x} = \frac{\left[\delta + \frac{4}{3}\nu_0\right]}{C_0} \mathbf{X}. \tag{2.19}$$

The classical theory of sound (see Landau and Lifshitz 1987, chap. VIII) is governed by a wave equation, which may be written in dimensionless form as

$$\frac{\partial^2 \phi}{\partial T^2} = \nabla^2 \phi. \tag{2.20}$$

The time derivative on the right of (2.18) leads to a decay of the waves not present in the classical theory.

The decay of the amplitude of separable solutions of (2.18) is governed by a telegraph equation. To see this consider the propagation of plane monochromatic traveling waves (see Landau and Lifshitz 1987, p 253). We can solve the one-dimensional version of (2.18)

$$\frac{\partial^2 \phi}{\partial T^2} = \left(1 + \frac{\partial}{\partial T}\right) \frac{\partial^2 \phi}{\partial X^2} \tag{2.21}$$

by separation of variables, $\phi = F(T)G(X)$, where

$$\frac{F''}{F + F'} = \frac{G''}{G} = -k^2. \tag{2.22}$$

The function $F(T)$ satisfies a telegraph equation. If $k^2 > 4$, the solution is

$$\phi = \left(Ae^{-\omega_1 T} + B^{-\omega_2 T}\right) \cos(-kX + \alpha) \tag{2.23}$$

where A, B and α are undetermined constants and

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \frac{k^2}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \sqrt{k^4 - 4k^2} \\ -\sqrt{k^4 - 4k^2} \end{bmatrix} \tag{2.24}$$

The solution is a standing periodic wave with a decaying amplitude.

If $k^2 < 4$, the solution is

$$\phi = e^{-\frac{k^2}{2}T}\left[\begin{array}{c} A\cos\left(-kX - \frac{1}{2}(4k^2 - k^4)^{1/2}T + \alpha\right) \\ +B\cos\left(-kX + \frac{1}{2}(4k^2 - k^4)^{1/2}T + \alpha\right) \end{array}\right] \tag{2.25}$$

represents decaying waves propagating to the left and right.

Travelling plane wave solutions which are periodic in T and grow or decay in X are also easily derived by separating variables. The travelling wave

$$\begin{aligned} \phi = & Ae^{-k_1 X}\cos(k_2 X - \omega T + \alpha) \\ & +Be^{k_1 X}\cos(-k_2 X - \omega T + \alpha) \end{aligned} \tag{2.26}$$

is such a solution provided that

$$k_1 = \frac{1}{\sqrt{2}}\frac{\omega^2}{\left[p + (p^2 + \omega^2 p^2)^{1/2}\right]^{1/2}}, \qquad k_2 = \frac{1}{\sqrt{2}}\frac{\left[p + (p^2 + \omega^2 p^2)^{1/2}\right]^{1/2}}{p}\omega,$$

where $p = 1 + \omega^2$.

The separation of variables for plane waves may be greatly generalized by considering solutions of (2.18) of the form $\phi = F(T)G(X, Y, Z)$ leading to a separation of variables like (2.22) in the form

$$\frac{F''}{F + F'} = \frac{\nabla^2 G}{G} = -k^2 \tag{2.27}$$

where $F(T)$ satisfies the same telegraph equation as for plane waves.

3. Concluding remarks

All of the potential flow solutions which perturb the state of rest of an inviscid compressible fluid can be considered for the effects of viscosity using the potential flow equations for viscous compressible flows derived here. Under ordinary circumstances viscous and relaxation effects will be negligible. In problems of high frequency ultrasound in liquids however, the effects of viscosity can be important, even dominant. The viscous effects which would enter into the study of stress induced cavitation (Joseph 1998) due to high frequency are two-fold: dissipative effects which are more or less described by a telegraph equation, and "anistropic" pressures associated with the viscous part of the stress tensor (2.13).

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