Studies of Viscous and Viscoelastic Potential Flows

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C. Project Description

The purpose of this proposal is to carry out the extension of mathematical studies of inviscid potential flow to viscous and viscoelastic potential flow. My claim is that inviscid potential flow is a special case of viscous potential flow in which the viscosity is put to zero and it is without merit even from the point of view of mathematical simplicity. Viscous potential flow is the potential flow solution of the Navier-Stokes equations which has all the properties of inviscid potential flow except that the viscous stresses do not in general vanish. Viscous and viscoelastic potential flows give rise to excellent physical results for flow with interfaces; for such flows the viscosity enters the analysis explicitly through the normal stress balance. Viscous potential flow has zero vorticity and it is an approximation which will certainly fail when vorticity is important; however in the material to follow, I will try to show that if you like inviscid potential flow, viscous potential flow is better.

The first goal of the proposed research is to carry out the analysis of viscous and viscoelastic potential flow for the many interface problems which have been solved by inviscid potential flow. This has already been done for Rayleigh-Taylor instability of a viscous fluid and a viscoelastic fluid, for Kelvin-Helmholtz instability and for capillary instability. Viscous potential flow was applied to the problem of determining the rise velocity of a spherical cap bubble which was studied by Davies and Taylor (1950) using inviscid potential flow (see Batchelor 1967, pg. 475). The effects of viscosity which could not emerge from Taylor's inviscid analysis do arise very naturally, and in elementary explicit form, from analysis based on viscous potential flow. The results presented in these papers show that viscous and viscoelastic analysis of flow instability is very often close to the results of exact analysis and always better than the results of analysis based on inviscid potential flow.

The program of research we propose to follow is as follows:

(1) Extend the results of published works of quality based on inviscid potential flow to viscous and viscoelastic potential flows.

(2) Understand, characterize and evaluate the differences between dissipation calculations carried out on the velocity field given by inviscid potential flow and viscous potential flow.

Levich (1962) computed the relation of the rise velocity to the drag on a rising gas bubble by computing the viscous dissipation the potential flow of an inviscid fluid outside a moving sphere. Lamb (1924) computed the rate of decay of a free wave on an inviscid fluid by evaluating the dissipation. These dissipation calculations are approximations to the full Navier-Stokes equations at high Reynolds numbers under conditions in which potential flow of an inviscid fluid is believed to be close to real flows. The relation of these dissipation calculations using potential flow to viscous potential flow is in need of clarification.

(3) Apply viscous potential flow to problems in which viscous stress in irrotational flows could be important. Cavitation of liquids at the final stage of capillary collapse, super-cavitation in atomizers and cavitation due to ultrasound are three problems in which viscous extensional stresses, which may be calculated using viscous potential flow, are important.

1 Viscous and viscoelastic potential flow

Potential flow $\mathbf{u} = \nabla\phi$ are solutions of the Navier-Stokes equations for viscous incompressible fluids. The viscous term $\mu\nabla^2\mathbf{u} = \mu\nabla\nabla^2\phi$ vanishes, but the viscous contribution to the stress in an incompressible fluid (Stokes 1850)

$$T_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = -p\delta_{ij} + 2\mu\frac{\partial^2\phi}{\partial x_i \partial x_j} \tag{1}$$

does not vanish in general. Not all models of viscoelastic fluids admit a potential flow solution; the curl of divergence of the extra stress must vanish. Potential flows of incompressible fluids admit a pressure (Bernoulli) equation when the divergence of the stress is a gradient as in inviscid fluids, viscous fluids, linear viscoelastic fluids and second order fluids (for which a term proportional to the square of the velocity gradient called a viscoelastic pressure appears). All of the classical results for inviscid potential flows hold for viscous potential flow with the caveat that the viscous stresses are not generally zero. The differences between inviscid and viscous and viscoelastic potential flow together with a review of the literature prior to 1994 are discussed by Joseph and Liao (1994a,b).

Potential flows will not generally satisfy boundary conditions which are associated with the requirement that the tangential component of velocity and the shear stress be continuous across the interface separating the fluid from a solid or another fluid. The velocity and pressure in viscous potential flow is the same as inviscid potential flow when fluid-fluid interfaces or free surfaces are not present.

The viscosity enters explicitly into the problem formulation for interface problems through the viscous term in the normal stress balance across the interface. Viscous potential flow analysis gives good approximations to fully viscous flows in cases where the shear from the gas flow is negligible; the Rayleigh-Plesset bubble is a potential flow which satisfies the Navier-Stokes equations and all the interface conditions. Joseph, Belanger and Beavers (1999) constructed a viscous potential flow of the Rayleigh-Taylor instability which is almost indistinguishable from the exact fully viscous analysis. Joseph, Beavers and Funada (2002) constructed a viscoelastic potential flow analysis for the Rayleigh-Taylor instability of an Oldroyd-B model which is also in very good agreement with the unapproximated solution. The two papers just mentioned were applied to experiments on drop breakup at very high Weber numbers and give rise to satisfying agreements.

Funada and Joseph (2001) gave a viscous potential flow analysis of Kelvin-Helmholtz instability in a channel. There is no exact solution for the linearized viscous equations for this problem but a number of approximate solutions have been given. Mata, Pereyra, Trallero and Joseph (2002) compared these theories with experiments. The theories do not agree with each other and only the viscous potential flow solution of Funada and Joseph agrees with the experiments.

Funada and Joseph (2002a) gave a viscous potential flow analysis of capillary instability. Results of linearized analysis based on potential flow of a viscous and inviscid fluid were compared with the unapproximated normal mode analysis of the linearized Navier-Stokes equations. The growth rates for the inviscid fluid are largest, the growth rates of the fully viscous problems are smallest and those of viscous potential flow are between. The growth rates of the fully viscous fluid analysis and viscous potential flow are uniformly in good agreement. The results from all three theories converge when a Reynolds number $\gamma D\rho_e/\rho^2_e$ based on the velocity γ/γ_e of capillary collapse is large (γ, D, ρ_e, μ_e) = (surface tension, diameter, density, viscosity). The convergence results apply to two liquids as well as to liquid and gas. Funada and Joseph (2002b) did the same type of analysis of capillary instability of the Maxwell model of a viscoelastic fluid. The results are similar to those for viscous potential flow.

In a recent paper Joseph (2002) applied the theory of viscous potential flow to the problem of finding the rise velocity U of a spherical cap bubble (Davies and Taylor 1950, Batchelor 1967). The rise velocity is given by

$$\frac{U}{\sqrt{gD}} = -\frac{8}{3}\frac{\nu(1+8s)}{\sqrt{gD^3}} + \frac{\sqrt{2}}{3}\left[1-2s-\frac{16s\sigma}{\rho g D^2}+\frac{32\nu^2}{gD^3}(1+8s)^2\right]^{1/2} \tag{2}$$

where $R = D/2$ is the radius of the cap, ρ and ν are the density and kinematic viscosity of the liquid, σ is surface tension and $s = r''(0)/D$ is the deviation of the free surface

$$r(\theta) = R + \tfrac{1}{2} r''(0)\theta^2 = R(1 + s\theta^2) \tag{3}$$

from perfect sphericity $r(\theta) = R$ near the stagnation point $s = 0$. The bubble nose is more pointed when $s < 0$ and blunted when $s > 0$. A more pointed bubble increases the rise velocity; the blunter bubble rises slower.

The Davies-Taylor (1950) result

$$U = \frac{\sqrt{2}}{3}\sqrt{gD}$$

arises when all other effects vanish; if s alone is zero,

$$\frac{U}{\sqrt{gD}} = -\frac{8}{3}\frac{\nu}{\sqrt{gD^3}} + \frac{\sqrt{2}}{3}\left[1+\frac{32\nu^2}{gD^3}\right]^{1/2} \tag{4}$$

showing that viscosity slows the rise velocity.

- ***Topic 1: Extend the results of published works of quality based on inviscid potential flow to viscous and viscoelastic potential flows.***

The entry "potential flow" in Google's internet search engine gives rise to 2,230,000 hits. None of these, except possibly for the few works already cited, would be for viscous potential flow. Updating of even a tiny fraction of the papers on inviscid potential to viscous potential flow is at least a lifetime of work, even for a younger man. At present, we are carrying out analysis of spatial, absolute and convective instability of liquid jets using viscous potential flow The analysis using the method Briggs (1964) and the singularity calculation of pinch mentioned on page 275 of the book by Schmid and Henningson (2001). If $D(k,\omega) = 0$ is the dispersion relation, then the singularities (pinch points) in the k plane must satisfy $\partial D\ D(k,\omega)/\partial k = 0$ where k and ω are both complex valued. These singularities allow us to distinguish the conditions under which the flow is absolutely or convectively unstable. In the liquid jet case we get an explicit dispersion relation of the inviscid analysis $D(k,\omega) = 0$ and the generalization of the inviscid analysis to viscous potential flow gives then explicit results about the effects of viscosity. The results are gratifying since the criteria distinguishing absolute from convective stability is nearly the same as criteria derived by Lin and Lian (1989) who studied this problem for a viscous liquid jet without assuming potential flow. The same sort of explicit analysis based on viscous potential flow works perfectly well for two liquids, as well as for liquid and gas.

A bewildering number of excellent free surface problems solved by inviscid potential flow can be extended easily to include the effects of viscosity using viscous potential flow.

The entry "free surface flow" on Google gives rise to 294,000 hits with a large number of problems solved by inviscid potential flow such as, deformation of drops and bubbles in uniform flow, cavitation problems, jets rising and falling under gravity, jets ejected from square or elliptical orifices, weir flows, resonantly interacting water waves and many kinds of interface stability problems, to name a few.

- ***Topic 2: Understand, characterize and evaluate the differences between dissipation approximations and viscous potential flow.***

It is desirable to draw attention to the fact that different predictions arise for the same problems comparing viscous potential flow with dissipation approximations based on potential flow. This is explained below; we show how different predictions of the rise velocity of gas bubbles in liquids and the decay rate of gravity on water.

Consider the case of a liquid and gas in which the gas is passive having no dynamic consequence on the liquid. Let V the volume of the liquid and A is its boundary. The Navier-Stokes equations for the liquid are

$$\rho\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\nabla p + \mu\,\mathrm{div}\,\mathbf{D},\ \ \mathrm{div}\,\mathbf{u} = 0 \tag{5}$$

The mechanical energy equation for (5) is

$$\frac{\rho}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_V \mathbf{u}^2 dV = \int_A \mathbf{u}\cdot(\mathbf{T}\cdot\mathbf{n})\,\mathrm{d}A - \int_V 2\mu\,\mathbf{D}:\mathbf{D}\,\mathrm{d}V \tag{6}$$

where

$$\mathbf{T} = -p\mathbf{1} + 2\mu\,\mathbf{D}[\mathbf{u}] \tag{7}$$

is the stress and $\mathbf{D}[\mathbf{u}]$ is the rate of strain.

In the case of the gas bubble it is assumed that the bubble is in steady flow. If it were a solid then every point on the sphere would move with the same velocity

$$\mathbf{u} = \mathbf{e}_x U \quad \text{for } \mathbf{x} \text{ on } A. \tag{8}$$

Then

$$UD = 2\mu \int_V \mathbf{D} : \mathbf{D} \,\mathrm{d}V \tag{9}$$

where

$$\mathrm{D} = \int_A \mathbf{e}_x \cdot \mathbf{T} \cdot \mathbf{n} \,\mathrm{d}a \tag{10}$$

Equation (9) was used by Levich (1962) but (8) does not hold on A in the case of gas bubble. If this approximation is employed, then

$$DU = 2\mu \int_V \mathbf{D} : \mathbf{D} \,\mathrm{d}V = 2\mu \int_V \frac{\partial^2 \phi}{\partial x_i \partial x_j} \frac{\partial^2 \phi}{\partial x_i \partial x_j} \mathrm{d}V \tag{11}$$

where

$$\mathbf{u} = \nabla \phi,\ \phi = -\frac{1}{2} U a^3 \frac{\cos\theta}{r^2} \tag{12}$$

and

$$D = 12\pi a \mu \mathrm{U},\quad C_D = 48/R \tag{13}$$

Analysis of the rising bubble based on viscous potential flow is the same as for inviscid potential flow except that the viscous contribution to the normal stress balance must be included. This viscous contribution would lead to a distortion of the spherical shape of the bubble which could be computed as a perturbation from the spherical shape in powers of γ^{-1} where γ is the surface tension. Moore (1959) applied the normal stress boundary condition to a passive spherical bubble and found using (12)

$$\mathbf{T}_{rr} = -p + 2\mu \frac{\partial u}{\partial r} = p_I - 6\left(\frac{\mu U}{a}\right)\cos\theta \tag{14}$$

where p_I is the pressure from the potential flow solution. Counting the tangential stress of the potential flow on the bubble surface as zero, he computed

$$D = 8\pi\mu U a,\ C_D = 32/\mathrm{R} \tag{15}$$

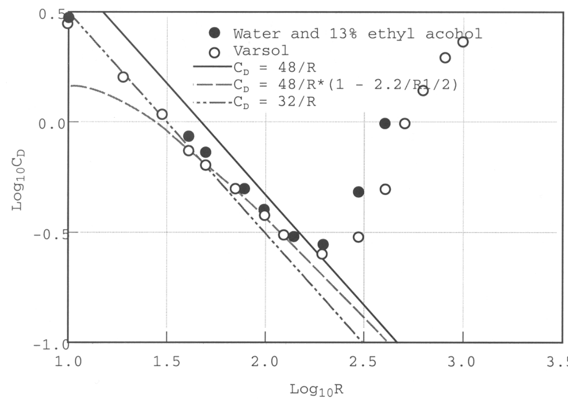
In a later paper, Moore (1963) carried out a boundary layer analysis at the surface of the bubble and found that

$$C_D = \frac{48}{R}\left\{1 - \frac{2.2}{\sqrt{R}} + \cdots\right\} \tag{16}$$

The leading order agrees with the Levich formula.

This problem has been reviewed by Batchelor (1967) who has compared (13) and (16) with experimental data (see our Figure 1). We added C_D = 32/R as —··— ; it is in rather better agreement with the data than (13) or (16).

Figure 1. (after Batchelor 1967.) The drag coefficient of gas bubbles rising through liquids. The points for two particular liquids are taken from experimental curves given by Haberman and Morton (1953). The line C_D = 32/R was added by me.



Lamb (1924, p. 624) considered the effect of the viscous dissipation of a free traveling wave given by the potential

$$\phi = ace^{ky} \cos k\,(x\text{-}ct) \tag{17}$$

and finds that the mean value of the dissipation per unit area is given

$$2\mu k^3 a^2 c^2 \tag{18}$$

"The kinetic energy per unit area is ¼ $\rho k a^2 c^2$, and the total energy (kinetic plus potential) is therefore double of this. Hence in the absence of surface forces

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\rho k c^2 a^2\right) = -2\mu k^3 a^2 c^2\,, \tag{19}$$

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -2\nu k^2 a, \qquad a = a_0 e^{-2\nu k^2 t}\,. \text{"} \tag{20}$$

Equation (20) gives the rate of decay of a free wave on an inviscid fluid due to viscosity.

It is convenient to interpret Lamb's results in terms of gravity waves for which

$$c = \sqrt{\frac{g}{k}} \tag{21}$$

When a gravity term $-\rho g$ is added to the right side of (5) the energy equation (6) may be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon + \wp) = \int_0^{2\pi/k} \mathbf{u}\cdot\mathbf{T}\cdot\mathbf{n} - \int_{-\infty}^{n}\int_0^{2\pi/k} 2\mu\,\mathbf{D}:\mathbf{D}\,\mathrm{d}y\,\mathrm{d}x \tag{22}$$

where ε is the kinetic energy and $\wp$ the potential energy

$$\wp = \int_0^{2\pi/k} \rho g \eta^2\,\mathrm{d}x \tag{23}$$

where $z - \eta(x,t) = 0$ (see Joseph 1976, p. 250). For the free motion of an inviscid potential the stress traction term will vanish and the left side of (22) can be computed in the linear case as in (19). The stress traction term should not be neglected for viscous potential flow.

An analysis of the stability of gravity waves using viscous potential is embedded in the analysis of Kelvin-Helmholtz instability by Funada and Joseph (2001). A free wave is not stable, it must decay but at half the rate given by Lamb's dissipation calculation. In the analysis of linear stability of gravity waves based on viscous potential flow $\mathbf{u} = \nabla\phi$, $\nabla^2\phi = 0$ we find, after eliminating the pressure in the normal stress balance, that

$$\frac{\partial\phi}{\partial t} + g\eta + 2\nu\frac{\partial^2\phi}{\partial y^2} = 0 \tag{24}$$

and from the kinematic condition for $y = \eta$ we get

$$\frac{\partial\phi}{\partial y} = \frac{\partial\eta}{\partial t} \tag{25}$$

on $y = 0$. After eliminating η in (24) using (25) and applying normal modes (17) we find

$$c^2 + 2\nu ick^2 - g/k = 0 \tag{26}$$

or

$$c = -i\nu k \pm \sqrt{\frac{g}{k} - \nu^2 k^2} \tag{27}$$

Hence the normal solution is proportional to

$$e^{-\nu k^2 t} e^{ik\left(x \pm t\sqrt{\frac{g}{k} - \nu^2 k^2}\right)} \tag{28}$$

The amplitude of the wave decays at a rate

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \nu k^2 a\,, \tag{29}$$

one-half of the rate given by (20). The wave speed c is given by

$$c = \sqrt{\frac{g}{k} - \nu^2 k^2}\,, \tag{30}$$

which is slower than $\sqrt{g/k}$ for $k^3 < g/\nu^2$. For very large values of k, short standing waves do not propagate but simply decay at a rate given

$$a = a_0 \exp\left\{-\frac{1}{2}\frac{g}{\nu k}t\right\}. \tag{31}$$

As far as I know there are no measurements on the decay rate of gravity waves due to viscosity.

- ***Topic 3. Compute the effects of viscosity on sound waves using viscous potential flow.***

(i) Potential flow solutions of the Navier-Stokes equations for viscous compressible fluids

Potential flows are not in general solutions of the compressible Navier-Stokes equations. To have such solutions it is necessary to show that curl $\mathbf{u} = 0$ is a solution (see Joseph and Liao

Equation numbers in Topics 3,4 need to be updated! --dave

1994a) of the vorticity equation. The gradients of density and viscosity which are spoilers for the general vorticity equation do not enter into the equations which perturb the state of rest with uniform pressure p_0 and density ρ_0.

The stress for a compressible viscous fluid is given by

$$T_{ij} = -\left(p + \frac{2}{3}\mu \operatorname{div} \mathbf{u}\right)\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{5}$$

Here, the second coefficient of viscosity is selected so that $T_{ii} = -3p$. (The results to follow can be worked for arbitrary choices of the second coefficient of viscosity.)

The equations of motion are given by

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = \operatorname{div} \mathbf{T} \tag{6}$$

together with

$$\frac{\partial \rho}{\partial t} + \mathbf{u}\cdot\nabla\rho + \rho \operatorname{div} \mathbf{u} = 0\,. \tag{7}$$

To study acoustic propagation, these equations are linearized; putting

$$[\mathbf{u}, p, \rho] = [\mathbf{u}', p_0 + p', \rho_0 + \rho'] \tag{8}$$

where , p and ρ are small quantities, we get

$$T_{ij} = -\left(p_0 + p' + \frac{2}{3}\mu_0 \operatorname{div} \mathbf{u}'\right)\delta_{ij} + \mu_0\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right) \tag{9}$$

$$\rho_0 \frac{\partial \mathbf{u}'}{\partial t} = -\nabla p' + \mu_0\left(\nabla^2\mathbf{u}' + \frac{1}{3}\nabla \operatorname{div} \mathbf{u}'\right) \tag{10}$$

$$\frac{\partial \rho'}{\partial t} + \rho_0 \operatorname{div} \mathbf{u}' = 0 \tag{11}$$

where p_0, ρ_0 and μ_0 are constants. For acoustic problems, we assume that a small change in ρ induces small changes in p by fast adiabatic processes; hence

$$p' = C_0^2 \rho' \tag{12}$$

where C_0 is the speed of sound.

Forming now the curl of (10), we find that

$$\rho_0 \frac{\partial \boldsymbol{\zeta}}{\partial t} = \mu_0 \nabla^2 \boldsymbol{\zeta}, \;\; \boldsymbol{\zeta} = \operatorname{curl} \mathbf{u}'\,. \tag{13}$$

Hence $\boldsymbol{\zeta} = 0$, is a solution of the vorticity equation and we may introduce a potential

$$\mathbf{u}' = \nabla\phi. \tag{14}$$

Combining next (14) and (10), we get

Equation numbers in Topics 3,4 need to be updated! --dave

$$\nabla\left[\rho_0\frac{\partial\phi}{\partial t}+p'-\frac{4}{3}\mu_0\nabla^2\phi\right]=0\,. \tag{15}$$

The quantity in the bracket is equal to an arbitrary function of the time which may be absorbed in ϕ.

A viscosity dependent Bernoulli equation

$$\rho_0\frac{\partial\phi}{\partial t}+p'-\frac{4}{3}\mu_0\nabla^2\phi=0 \tag{16}$$

is implied by (15). The stress (9) is given in terms of the potential ϕ by

$$T_{ij}=-\left(p_0-\rho_0\frac{\partial\phi}{\partial t}+2\mu_0\nabla^2\phi\right)\delta_{ij}+2\mu_0\frac{\partial^2\phi}{\partial x_i\partial x_j}\ . \tag{17}$$

To obtain the equation satisfied by the potential ϕ, we eliminate ρ' in (11) with p' using (12), then eliminate $\mathbf{u}'=\nabla\phi$ and p' in terms of ϕ using (16) to find

$$\frac{\partial^2\phi}{\partial t^2}=\left(C_0^2+\frac{4}{3}v_0\frac{\partial}{\partial t}\right)\nabla^2\phi \tag{18}$$

where the potential ϕ depends on the speed of sound and the kinematic viscosity $v_0=\mu_0/\rho_0$.

(ii) Sound waves

A dimensionless form for the potential equation (18)

$$\frac{\partial^2\phi}{\partial T^2}=\left(1+\frac{\partial}{\partial T}\right)\nabla^2\phi,\qquad \nabla^2\phi=\frac{\partial^2\phi}{\partial X^2}+\frac{\partial^2\phi}{\partial Y^2}+\frac{\partial^2\phi}{\partial Z^2} \tag{19}$$

arises from a change of variables

$$t=\frac{4v_0}{3C_0^2}T,\ \ \mathbf{x}=\frac{4}{3}\frac{v_0}{C_0}\mathbf{X}\,. \tag{20}$$

The classical theory of sound (see Landau and Lifshitz 1987, chap. VIII) is governed by a wave equation, which may be written in dimensionless form as

$$\frac{\partial^2\phi}{\partial T^2}=\nabla^2\phi\,. \tag{21}$$

The time derivative on the right of (19) leads to a decay of the waves not present in the classical theory. Many if not all of the results obtained with (21) may be redone, using (19).

(iii) Plane monochromatic travelling waves

This is the simplest problem of sound waves (see Landau and Lifshitz 1987, p 253). First we separate variables, inserting

$$\phi=F(T)\,G(X) \tag{22}$$

into the one-dimensional version of (19)

Equation numbers in Topics 3,4 need to be updated! --dave

$$\frac{\partial^2\phi}{\partial T^2}=\left(1+\frac{\partial}{\partial T}\right)\frac{\partial^2\phi}{\partial X^2}, \tag{23}$$

and find that

$$\frac{F''}{F+F'}=\frac{G''}{G}. \tag{24}$$

The only way that the function of T on the left could be equal to the function of X on the right is if both sides are constant. For periodic waves corresponding to (64.19) in Landau and Lifshitz, we put

$$G''=-k^2G. \tag{25}$$

Hence,

$$F''+k^2F'+k^2F=0. \tag{26}$$

Equation (26) is a telegraph equation leading to damped plane, travelling, monochromatic waves. Functions of the form $e^{-\omega T}$ are solutions of (26) if $\omega^2-\omega k^2+k^2=0$. This quadratic equation has two roots

$$\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}=\frac{k^2}{2}\begin{bmatrix}1\\ 1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}\sqrt{\Delta}\\ -\sqrt{\Delta}\end{bmatrix} \tag{27}$$

where $\Delta=k^4-4k^2$. If $\Delta>0$, ($k^2>4$), then ω_1 and ω_2 are both positive and

$$\phi=\left(Ae^{-\omega_1T}+Be^{-\omega_2T}\right)\cos(-kX+\alpha) \tag{28}$$

where A,B and α are undetermined constants. The solution is a standing periodic wave with a decaying amplitude.

If $\Delta<0$ ($k^2<4$), there are two complex roots given by (27) and

$$\phi=e^{-\frac{k^2}{2}T}\begin{bmatrix}A\cos\left(-kX-\frac{1}{2}\left(4k^2-k^4\right)^{1/2}T+\alpha\right)\\ +B\cos\left(-kX+\frac{1}{2}\left(4k^2-k^4\right)^{1/2}T+\alpha\right)\end{bmatrix} \tag{29}$$

represents decaying waves propagating to the left and right.

Traveling plane wave solutions which are periodic in T and grow or decay in X are also easily derived by separating variables. The traveling plane wave

$$\begin{aligned}\phi=&Ae^{-k_1X}\cos(k_2X-\omega T+\alpha)\\ &+Be^{k_1X}\cos(-k_2X-\omega T+\alpha)\end{aligned} \tag{30}$$

is such a solution provided that

$$k_1=\frac{1}{\sqrt{2}}\frac{\omega^2}{\left[p+\left(p^2+\omega^2p\right)^{1/2}\right]^{1/2}},\qquad k_2=\frac{1}{\sqrt{2}}\frac{\left[p+\left(p^2+\omega^2p\right)^{1/2}\right]^{1/2}}{p}\omega,$$

Equation numbers in Topics 3,4 need to be updated! --dave

where $p = 1 + \omega^2$.

(vi) Separable solutions

The separation of variables (22) given for plane waves may be greatly generalized by considering solutions of (19) of the form

$$\phi = F(T)\, G(X,Y,Z) \tag{31}$$

leading to a separation of variables like (24) in the form

$$\frac{F''}{F+F'} = \frac{\nabla^2 G}{G} \tag{32}$$

where

$$\nabla^2 G = -k^2 G \tag{33}$$

and F satisfies (26).

For spherically symmetric waves,

$$\frac{\partial^2 \phi}{\partial T^2} = \left(1 + \frac{\partial}{\partial T}\right)\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial \phi}{\partial R}\right) \tag{34}$$

and, following Landau and Lifshitz (1987, pg. 269) we note that

$$\psi(R,T) = R\phi \tag{35}$$

satisfies equation (23) for plane waves with R replacing X. It follows then that the solutions (28), (29) and (30) hold for spherically symmetric waves when X is replaced with R and $\phi(X,T)$ with $R\phi(R,T)$.

All of the potential flow solutions which perturb the state of rest of an inviscid compressible fluid can be considered for the effects of viscosity using the potential flow equations for viscous compressible flows derived here. Sound propagation due to multiple sources and viscous effects in ultrasound are among the application areas to which these equations may apply.

- ***Topic 4: Apply viscous potential flow to problems in which stresses computed on irrotational flow could be important.***

  Two such problems are proposed.

  (i) Stress induced cavitation as the final stage of capillary collapse and rupture.

  (ii) Stress induced cavitation in atomizers.

All of these problems look to effects of cavitation and are based on a theory of stress induced cavitation put forward by Joseph (1995, 1998). The theory is based on the observation that the pressure in a liquid is the mean normal stress and the liquid cannot average its stresses; the state of the stress at a point is relevant and the liquid will break or cavitate under tension. It is necessary to look at the state of stress in principal coordinates to compare the maximum tension with the breaking strength or cavitation threshold. A cavity will open in the direction of maximum tensile stress, which is 45° from the plane of shearing in pure shear of Newtonian fluid.

The computation of internal stresses due to motion should be considered in all problems in which cavitation is an issue. The calculation of principal stress at each point in a flow is greatly simplified when the velocity field is given by a potential and the stress by equation (1). This simplification has value for flows in which the regions of vorticity creation are confined to small layers near solid boundaries. Batchelor (1967, pg 398) said that results of irrotational flow "…may be applied directly to cases of flow at large Reynolds number in which boundary-layers separation does not occur (which would include slender bodies moving parallel to their length, and bodies of arbitrary shape accelerating from rest or executing translational or rotational oscillation of small amplitude about a fixed position…" Flows from reservoirs of fluid at rest in nozzles are irrotational near the nozzle entrance before the boundary layers build up. Batchelor (pg 276) notes that "… a body of inviscid fluid in irrotational continues to move irrotationally." I like better the statement on page 277 that, "The conditions under which irrotational motion remains irrotational are those for which Kelvin's circulation theorem is valid" because this theorem is valid for the irrotational flow of a viscous fluid. Potential flow simplifies the search for the relevant physics by simplifying the mathematics even in cases where more should be done.

(i) Stress induced cavitation as the final stage of capillary collapse and rupture

Cavitation will occur in pure extension when the extensional stress is large enough, at high rates of extension. Lundgren and Joseph (1998[1]) looked this idea to elucidate the mechanism of rupture of a liquid cylinder under capillary collapse. They analyzed the breakup of a capillary filament as a viscous potential flow near a stagnation point on the centerline of the filament towards which the surface collapses under the action of surface tension forces. They found that the neck is of parabolic shape and its radius collapses to zero in a finite time. During the collapse the tensile stress due to viscosity increases in value until at a certain finite radius, which is about 1.5 microns for water in air, the stress in the throat passes into tension, presumably inducing cavitation there. The problem of capillary collapse or "pinching" has recently seen a burst of interest possibly due to the discovery of several similarity solutions (Eggers 1993, 1997; Papageogiou 1995) and others reviewed in the paper of McKinley and Tripathi (2000). These authors are not interested in the physics of rupture or breakup by cavitation and they do not compute stresses. All of the above mentioned authors find that capillary radius decreases to zero linearly in time, but the rate of collapse differs from author to author. McKinley and Tripathi (2000) write the formula

$$R_{mid}{}^{(t)} = R_1 - \frac{2X-1}{6}\frac{\gamma}{\eta}t \tag{32}$$

for the neck radius of the collapsing capillary in the stage of final decay as t increases to t^* where $R_{mid}{}^{(t^*)} = 0$. They give the X obtained by different authors in their table 1, but without the value $X = 2$ obtained by Lundgren and Joseph for viscous potential flow, who give the fastest decay. Eggers (1993, 1997) obtained $X = 0.5912$ and Papageoriou (1995) obtained $X = 0.719$. The solutions of the two authors last named have vorticity; Papageriou's solution has no inertia. Lundgren and Joseph found that the Reynolds number Re = $R_{CR}\,\gamma/2\upsilon\eta$ based on velocity $\upsilon = \gamma/\eta$ of capillary collapse at the point of capillary collapse where

R_{CR} = 1 micron is about 55.

[1] This is available in PDF format from *http://www.aem.umn.edu/people/faculty/joseph/archive/docs/capillary.pdf.*

Essential issues about the final collapse are suggested in the following citation of McKinley and Tripathi (2000):

> "Very close to the breakup event, Brenner [Lister and Stone] (1996) and Eggers (1997) note that the inertial effects can no longer be neglected in the fluid, since the local rate of extensional deformation $\left[\varepsilon \approx \left(R_{mid}\right)^{-1}\partial R_{mid} / \partial t\right]$ diverges. In this region one should thus expect the solution given in Eq. (9) to cross over from the inertialess similarity solution with X = 0.7127, to the universal form discovered by Eggers with X = 0.5912…"

McKinley and Tripathi appear to believe that the final decay ought to be described by a similarity solution. However the idea of a cross-over implies that in the region of cross-over there is another non-similarity solution. Maybe the final decay does not go to Eggers solution.

In the state of final collapse, the extensional stress $\eta\partial u/\partial x$ leading to tension gets very large but the capillary pressure due to the decreasing radius of the jet leads to compression. We find that the sum of the two effects in the similarity solutions do not lead to tension but the viscous potential flow solution does lead to tension.

The solution of Lundgren and Joseph (1998) is local at the neck and it is not rigorous; it rests on several assumptions, like stagnation point flow in the neck that need not be generated by a global viscous potential flow solution. The existence and properties of global potential solutions of the Navier-Stokes equations in the nonlinear case is an open question and the capillary collapse problem is just one realization. Viscous potential flow works well in the linear case and it ought to be studied for nonlinear problems.

We propose to apply viscous potential flow to the capillary collapse problem using the nonlinear case using numerical methods. We would look for capillary collapse on a periodic domain using a high resolution potential flow solver which would allow us to monitor the extensional stress at the final collapse, together with level set methods to resolve the interface conditions.

Studies of nonlinear problems based on viscous potential flow are necessary for the further evolution of this subject.

(ii) Stress induced cavitation of liquids in atomizers

We propose to use potential flow to look at stress induced cavitation in supercavitating nozzles. Supercavitation is a name introduced by Knapp, Daily and Hammitt (1970) for geometry induced cavitation which collapses away from the object that initiated it. Chaves, Knapp, Kubitzek, Obermeier and Schneider (1995) note that "Above an injection pressure threshold that depends on the nozzle geometry and chamber pressure, cavitation appears at the sharp inlet corner of the nozzle. With increasing injection pressure the cavitation reaches the nozzle exit (supercavitation). Reitz and Bracco (1982) identified four regimes of jet breakup; at the highest injection pressures the jet to drop size is much smaller than the jet diameter with breakup observed already at the nozzle exit. Bergwerk (1959) had observed cavitation in nozzles, starting at the nozzle entrance. Reitz and Bracco (1982) proposed that cavitation in the nozzle might be a mechanism for atomization. The observations of Soteriou, Andrews and Smith (1995) as well as Chaves, et al (1995) are consistent with hypothesis that the atomization is associated with supercavitation. Soteriou *et al* (1995) note that "The cavitating region consists of an opaque, creamy white foam…" which at one stage "…forms a ring close to the top of the hole."

The mechanism for supercavitation is not understood; it is believed to be associated with boundary layer separation.

We propose to explore the idea that cavitation at the exit ring is stress induced and to calculate the stress using potential flow for flow through an orifice under inlet conditions used for atomizers.

References need to be updated! --dave

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