

EQUATIONS FOR PROPPANT TRANSPORT

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1. Fracture Surface

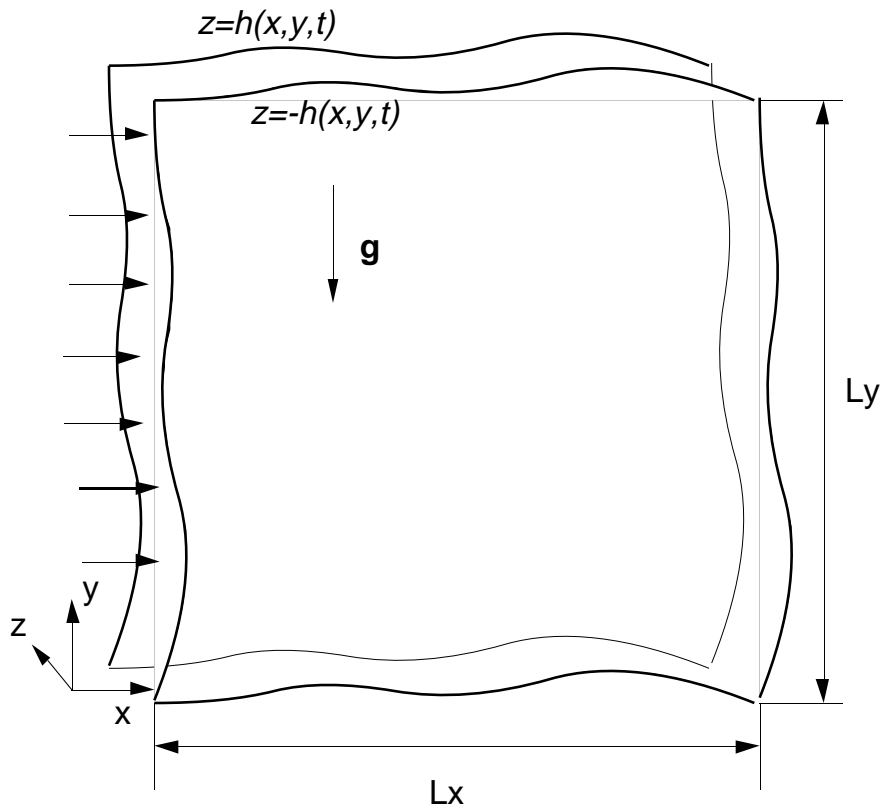


Figure 1. Flow of solid-liquid mixture through a fracture.

We consider flows of a solid-liquid mixture through a fracture as shown in Figure 1. The local width of the fracture is $2h(x,y,t)$. The fracture width is much smaller than the length scale in the other two dimensions $h/L_x \ll 1$ and $h/L_y \ll 1$, and the variation of the width is usually gentle, $(\partial h/\partial x)^2 + (\partial h/\partial y)^2 \ll 1$. A three dimensional orthogonal coordinate system can be chosen by locating (x,y) on the mid-surface, and z normal to the fracture surfaces, as shown in figure 1.

The gravity field is in the negative y direction and the flow is driven by an externally applied pressure gradient in the x direction. The velocity component in the z direction is normally much smaller than those in the other two directions, and is caused by the liquid leak-off at the fracture surfaces and by the spatial and temporal variation of the fracture surface. We assume that the flow is symmetric with respect to the middle plane $z = 0$.

2. Mass Conservation

We start from the mass conservation equations for multiphase flows:

$$\frac{\partial c}{\partial t} + \nabla \cdot (c\mathbf{u}_p) = 0, \quad (1)$$

$$\frac{\partial(1-c)}{\partial t} + \nabla \cdot [(1-c)\mathbf{u}_f] = 0 \quad (2)$$

where $c = c(x, y, z, t)$ is the solid volume concentration; \mathbf{u}_p and \mathbf{u}_f are the volume or ensemble averaged solid and fluid velocities, respectively. Adding (1) and (2), we have an equation for the composite velocity for the solid and fluid mixture,

$$\nabla \cdot \mathbf{u}_m = 0, \quad (3)$$

where

$$\mathbf{u}_m = c\mathbf{u}_p + (1-c)\mathbf{u}_f. \quad (4)$$

We can integrate the equation (1) across the narrow slot from middle surface $z = 0$ to one of the fracture surface $z = h(x, y, t)$, as shown in figure 1 (due to the symmetry, this integral is half the integral from $z = -h$ to $z = h$),

$$\int_0^h \frac{\partial c}{\partial t} dz + \int_0^h \left(\frac{\partial cu_p}{\partial x} + \frac{\partial cv_p}{\partial y} + \frac{\partial cw_p}{\partial z} \right) dz = 0. \quad (5)$$

Since

$$\begin{aligned} \int_0^h \frac{\partial c}{\partial t} dz &= \frac{\partial}{\partial t} \int_0^h c dz - c(x, y, h, t) \frac{\partial h}{\partial t}, \\ \int_0^h \frac{\partial cu_p}{\partial x} dz &= \frac{\partial}{\partial x} \int_0^h cu_p dz - (cu_p)(x, y, h, t) \frac{\partial h}{\partial x}, \\ \int_0^h \frac{\partial cv_p}{\partial y} dz &= \frac{\partial}{\partial y} \int_0^h cv_p dz - (cv_p)(x, y, h, t) \frac{\partial h}{\partial y}, \\ \int_0^h \frac{\partial cw_p}{\partial z} dz &= (cw_p)(x, y, h, t), \end{aligned} \quad (6)$$

we can rewrite (5) into,

$$\begin{aligned} & \frac{\partial}{\partial t} \int_0^h cz \, dz + \frac{\partial}{\partial x} \int_0^h cu_p \, dz + \frac{\partial}{\partial y} \int_0^h cv_p \, dz \\ & -c(x, y, h, t) \left[\frac{\partial h}{\partial t} + u_p \frac{\partial h}{\partial x} + v_p \frac{\partial h}{\partial y} - w_p \right]_{z=h} = 0 \end{aligned} \quad (7)$$

Notice that the kinematic condition for the surface $z = h(x, y, t)$ gives

$$w_p = \frac{\partial h}{\partial t} + u_p \frac{\partial h}{\partial x} + v_p \frac{\partial h}{\partial y}, \quad (8)$$

if the local no-slip conditions are true for the average solid velocities at the surface. Therefore, the equation (7) reduces to

$$\frac{\partial}{\partial t} \int_0^h cz \, dz + \frac{\partial}{\partial x} \int_0^h cu_p \, dz + \frac{\partial}{\partial y} \int_0^h cv_p \, dz = 0. \quad (9)$$

Or, we can write it into a different form,

$$\frac{\partial Ch}{\partial t} + \nabla \cdot \mathbf{Q}_p = 0 \quad (10)$$

where

$$\begin{aligned} C &= \frac{1}{h} \int_0^h cz \, dz \\ \mathbf{Q}_p &= \int_0^h c \mathbf{u}_p \, dz \stackrel{def}{=} hC(U_s \mathbf{i} + V_s \mathbf{j}) \end{aligned} \quad (11)$$

Please note that the mean fluxes (velocities) U_s and V_s are introduced. They are defined as the mean solid fluxes between the fracture surfaces. They are different from the mean solid average velocities.

Similarly, for the fluid velocity, we can integrate equation (2) to get

$$\frac{\partial(1-C)h}{\partial t} + \nabla \cdot \mathbf{Q}_f = -q_{lo} \quad (12)$$

where

$$\mathbf{Q}_f = \int_0^h (1-c)\mathbf{u}_f dz \stackrel{def}{=} h(1-C)(U_L\mathbf{i} + V_L\mathbf{j}). \quad (13)$$

and q_{lo} is the fluid leak-off velocity through the top ($z = h(x, y, t)$) wall. In deriving (12) we have to assume that the local no-slip conditions hold for the fluid velocity on the walls.

Equations (10) and (12) are two basic equations for the conservation of mass for the solid and fluid phases. We may add them together to generate an mass balance equation for the mixture,

$$\frac{\partial h}{\partial t} + \nabla \cdot \mathbf{Q}_m = -v_{lo} \quad (14)$$

where

$$\mathbf{Q}_m = \mathbf{Q}_p + \mathbf{Q}_f = \int_0^h [c\mathbf{u}_p + (1-c)\mathbf{u}_f] dz = \int_0^h \mathbf{u}_m dz \stackrel{def}{=} h(U_M\mathbf{i} + V_M\mathbf{j}). \quad (15)$$

Here the mean fluxes (velocities) for the mixture U_M and V_M coincide with the mean mixture velocities.

Explicitly we may organize the equations as

$$\frac{\partial hC}{\partial t} + \frac{\partial hCU_s}{\partial x} + \frac{\partial hCV_s}{\partial y} = 0 \quad (16)$$

$$\frac{\partial h(1-C)}{\partial t} + \frac{\partial h(1-C)U_L}{\partial x} + \frac{\partial h(1-C)V_L}{\partial y} + q_{lo} = 0 \quad (17)$$

and

$$\frac{\partial h}{\partial t} + \frac{\partial hU_M}{\partial x} + \frac{\partial hV_M}{\partial y} + q_{lo} = 0 \quad (18)$$

where

$$U_M = CU_s + (1-C)U_L \quad \text{and} \quad V_M = CV_s + (1-C)V_L \quad (19)$$

3. Momentum Equations

The momentum equations for the solid and fluid phases in multiphase flows normally take very complicated forms. Necessary constitutive relations are needed to model the interaction terms between the phases. However, in the momentum equation for the mixture of the solid and fluid, those interactions terms cancel. Especially, under the

lubrication approximations ($h/L_x \ll 1$ and $h/L_y \ll 1$, see figure 1), the momentum equation for the mixture reduces to

$$\begin{aligned}\frac{\partial}{\partial z} \left[\mu \frac{\partial u_m}{\partial z} \right] &= \frac{\partial p}{\partial x} \\ \frac{\partial}{\partial z} \left[\mu \frac{\partial v_m}{\partial z} \right] &= \frac{\partial p}{\partial y} - \rho_m g \\ \frac{\partial p}{\partial z} &= 0\end{aligned}\tag{20}$$

where the density of the composite mixture is

$$\rho_m = c\rho_{solid} + (1-c)\rho_{fluid},$$

ρ_{solid} and ρ_{fluid} are constant solid and fluid densities. In (20) we have assumed that the mixture behaves like a generalized Newtonian fluid whose viscosity is a function of both the solid concentration and the local shear rate, $\mu = \mu(c, \dot{\gamma})$. We may extend equation (20) to include some form of viscoelastic fluids, for example the second order fluid, by introducing a modified pressure using viscous potential flow theory. We may also define a mean composite density

$$\rho_M = \frac{1}{h} \int_0^h \rho_m dz = C\rho_{solid} + (1-C)\rho_{fluid} .\tag{21}$$

The third equation in (20) requires the pressure be a function of x and y only. The pressure variation in the y direction is caused by the gravity field and the gradient of the solid concentration. For a homogeneous mixture, the y component of the velocity v_m should be zero and we have static pressure profile in the y direction. The flow in the x direction is driven by the externally applied pressure gradient.

The solutions to (20) take the forms of

$$u_m = \frac{\partial p}{\partial x} \int_h^z \frac{z'}{\mu} dz' \tag{22}$$

$$v_m = \frac{\partial p}{\partial y} \int_h^z \frac{z'}{\mu} dz' - g \int_h^z \frac{1}{\mu} \left(\int_0^{z'} \rho_m dz'' \right) dz' .\tag{23}$$

If the mixture viscosity is independent of the shear rate and the solids are homogeneously distributed across the width of the fracture, (22) and (23) reduce to a simple form,

$$u_m = \frac{1}{2\mu(C)} \frac{\partial p}{\partial x} (z^2 - h^2) \quad (24)$$

$$v_m = \frac{1}{2\mu(C)} \left(\frac{\partial p}{\partial y} - \rho_M g \right) (z^2 - h^2). \quad (25)$$

Integrate the expression (24) and (25) across the fracture width, we have

$$\mathbf{Q}_m = \int_0^h \mathbf{u}_m dz = -\frac{h^3}{3\mu(C)} (\nabla p - \rho_M \mathbf{g}). \quad (26)$$

This expression will still be valid for generalized Newtonian fluids, however, the viscosity function $\mu(C)$ has to be replaced with some averaged value across the gap. For viscoelastic fluids, similar expression may exist, however, the definition of the viscosity function is not clear. Generally speaking, equation (26) holds if we assume that there is only one mesostructure of the solid-fluid two phase flow at any given set of flow conditions, and the mesostructure is set up instantaneously. There are experimental evidence that the time scale for setting up of the mesostructure is much shorter than the time scale of the global flow.

The variation of the mixture viscosity with the solid concentration can be approximated by

$$\mu(C) = \mu(0) \left(1 - \frac{C}{C_m} \right)^{-m}. \quad (27)$$

where C_m is the maximum packing concentration. Equation (26) can be written in the component form,

$$CU_S + (1-C)U_L = U_M = -\frac{h^2}{12\mu(C)} \frac{\partial p}{\partial x}, \quad (28)$$

$$CV_S + (1-C)V_L = V_M = -\frac{h^2}{12\mu(C)} \left(\frac{\partial p}{\partial y} - \rho_M g \right). \quad (29)$$

Here we have four equations (16, 17, 28 and 29) for six unknowns, U_S, V_S, U_L, V_L, p, C . We need two more equations to close the system. These two equations can be derived from the momentum equations for the solid phase or the fluid phase, or, as an alternative, we can introduce correlations.

One type of correlations introduces the concept of slip velocity between the solid and the mixture,

$$\mathbf{u}_{slip} = \mathbf{u}_p - \mathbf{u}_m = (1 - c)(\mathbf{u}_p - \mathbf{u}_f). \quad (30)$$

This slip velocity occurs both in the direction of the flow (x-direction) and in the direction of the gravity (y-direction). In the flow direction the slip is due to the inertia of the particles and is quite small when the particle Reynolds number is not large. However, in the direction of gravity the slip velocity is mainly due to the particle sedimentation and can be modeled as

$$\mathbf{u}_{slip} = f(c) \frac{2}{9} \frac{(\rho_{solid} - \rho_m) a^2}{\mu(c)} \mathbf{g} = \frac{f(c)(1 - c)}{\mu(c)} \frac{2}{9} \Delta\rho a^2 \mathbf{g}, \quad (31)$$

where a is the effective particle radius and $\Delta\rho = \rho_{solid} - \rho_{fluid}$, $f(c)$ is a function representing the hindered settling of the particles. The most popular and widely used form of this function is the Richardson and Zaki correlation

$$f(c) = (1 - c)^{4.65}. \quad (32)$$

In deriving the expression (31), we have used the Stokes drag law on a single particle. For flows with large particle Reynolds number we could use the empirical drag coefficient. This slip velocity in (31) can be used to generate an equation for the solid flux in the flow,

$$\begin{aligned} \mathbf{Q}_p &= hC(U_s \mathbf{i} + V_s \mathbf{j}) = \int_0^h c \mathbf{u}_p dz = \int_0^h c (\mathbf{u}_m + \mathbf{u}_{slip}) dz \\ &= -(\nabla p - \rho_M \mathbf{g}) \frac{h^2}{12\mu(0)} G(C) hC + \frac{2}{9} \frac{\Delta\rho a^2 \mathbf{g}}{\mu(0)} F(C) hC \end{aligned} \quad (33)$$

where $F(C)$ and $G(C)$ are two functions whose forms depend on the particle distribution across the fracture width. Function $F(C)$ is related to the hindered settling function, while $G(C)$ is related to the hold-up in the flow. Due to the particle migration toward the center of the fracture width, the averaged particle velocity is larger than the averaged fluid velocity. Equation (33) can be written in the component form,

$$U_s = -\frac{h^2}{12\mu(0)} \frac{\partial p}{\partial x} G(C). \quad (34)$$

$$V_s = -\frac{h^2}{12\mu(0)}\left(\frac{\partial p}{\partial y} - \rho_M g\right)G(C) + \frac{2}{9}\frac{\Delta\rho a^2 g}{\mu(0)}F(C). \quad (35)$$

Now six equations (16, 17, 28, 29, 34 and 35) solve for six unknowns, U_s, V_s, U_L, V_L, p, C .

4. Fracture Growth Equation

The fracture width $h(x,y,t)$ is related to the pressure field by the equation,

$$h(x,y,t) = \iint_{fracture} \frac{(1-\nu^2)}{\Pi E} \frac{p(\mathbf{x}',t)}{(\mathbf{x}-\mathbf{x}')} d\mathbf{x}' \quad (34)$$