C. Project Description

The purpose of this proposal is to carry out the extension of mathematical studies of inviscid potential flow to viscous and viscoelastic potential flow. My claim is that inviscid potential flow is a special case of viscous potential flow in which the viscosity is put to zero and it is without merit even from the point of view of mathematical simplicity. Viscous potential flow is the potential flow solution of the Navier-Stokes equations which has all the properties of inviscid potential flow except that the viscous stresses do not in general vanish. Viscous and viscoelastic potential flows give rise to excellent physical results for flow with interfaces; for such flows the viscosity enters the analysis explicitly through the normal stress balance. Viscous potential flow has zero vorticity and it is an approximation which will certainly fail when vorticity is important; however in the material to follow, I will try to show that if you like inviscid potential flow, viscous potential flow is better.

The first goal of the proposed research is to carry out the analysis of viscous and viscoelastic potential flow for the many interface problems which have been solved by inviscid potential flow. This has already been done for Rayleigh-Taylor instability of a viscous fluid and a viscoelastic fluid, for Kelvin-Helmholtz instability and for capillary instability. The results presented in these studies show that viscous and viscoelastic analysis of flow instability is very often close to the results of exact analysis and always better than the results of analysis based on inviscid potential flow. Viscous potential flow was applied to the problem of determining the rise velocity of a spherical cap bubble which was studied by Davies and Taylor (1950) using inviscid potential flow (see Batchelor 1967, pg. 475). The effects of viscosity which could not emerge from Taylor's inviscid analysis do arise very naturally, and in elementary explicit form, from analysis based on viscous potential flow. Papers of these studies can be found at *http://www.aem.umn.edu/people/faculty/joseph/ViscousPotentialFlow/*.

The program of research we propose to follow is as follows:

(1) Extend the results of published works of quality based on inviscid potential flows to viscous and viscoelastic potential flows.

(2) Understand, characterize and evaluate the differences between dissipation calculations carried out on the velocity field given by inviscid potential flow and viscous potential flow.

Levich (1962) computed the relation of the rise velocity to the drag on a rising gas bubble by computing the viscous dissipation using the potential flow of an inviscid fluid outside a moving sphere. Lamb (1924) computed the rate of decay of a free wave on an inviscid fluid by evaluating the dissipation. These dissipation calculations are approximations to the full Navier-Stokes equations at high Reynolds numbers under conditions in which potential flow of an inviscid fluid is believed to be close to real flows. The relation of these dissipation calculations using inviscid potential flow to viscous potential flow is in need of clarification.

(3) Apply viscous potential flow to problems in which viscous stress in irrotational flows could be important. Cavitation of liquids at the final stage of capillary collapse, super-cavitation in atomizers and stress induced cavitation are three problems in which viscous extensional stresses, which may be calculated using viscous potential flow, are important.

C. 1. Viscous and viscoelastic potential flow

Potential flows $\mathbf{u} = \nabla\phi$ are solutions of the Navier-Stokes equations for viscous incompressible fluids. The viscous term $\mu\nabla^2\mathbf{u} = \mu\nabla\nabla^2\phi$ vanishes, but the viscous contribution to the stress in an incompressible fluid (Stokes 1850)

$$T_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = -p\delta_{ij} + 2\mu\frac{\partial^2\phi}{\partial x_i \partial x_j} \tag{1}$$

does not vanish in general. Not all models of viscoelastic fluids admit a potential flow solution; the curl of divergence of the extra stress must vanish. Potential flows of incompressible fluids admit a pressure (Bernoulli) equation when the divergence of the stress is a gradient as in inviscid fluids, viscous fluids, linear viscoelastic fluids and second order fluids (for which a term proportional to the square of the velocity gradient called a viscoelastic pressure appears). All of the classical results for inviscid potential flows hold for viscous potential flows with the caveat that the viscous stresses are not generally zero. The differences between inviscid and viscous and viscoelastic potential flow together with a review of the literature prior to 1994 are discussed by Joseph and Liao (1994a,b).

Potential flows will not generally satisfy boundary conditions which are associated with the requirement that the tangential component of velocity and the shear stress should be continuous across the interface separating the fluid from a solid or another fluid. The velocity and pressure in viscous incompressible potential flow are the same as in inviscid incompressible potential flow when fluid-fluid interfaces or free surfaces are not present.

The viscosity enters explicitly into the problem formulation for interface problems through the viscous term in the normal stress balance across the interface. Viscous potential flow analysis gives good approximations to fully viscous flows in cases where the shear from the gas flow is negligible; the Rayleigh-Plesset bubble is a potential flow which satisfies the Navier-Stokes equations and all the interface conditions. Joseph, Belanger and Beavers (1999) constructed a viscous potential flow of the Rayleigh-Taylor instability which is almost indistinguishable from the exact fully viscous analysis. Joseph, Beavers and Funada (2002) constructed a viscoelastic potential flow analysis for the Rayleigh-Taylor instability of an Oldroyd-B model fluid which is also in very good agreement with the unapproximated solution. The two papers just mentioned were applied to experiments on drop breakup at very high Weber numbers and gave rise to satisfying agreements.

Funada and Joseph (2001) gave a viscous potential flow analysis of Kelvin-Helmholtz instability in a channel. There is no exact solution for the linearized viscous equations for this problem but a number of approximate solutions have been given. Mata, Pereyra, Trallero and Joseph (2002) compared these theories with experiments. The theories do not agree with each other and only the viscous potential flow solution of Funada and Joseph agrees with the experiments.

Funada and Joseph (2002a) gave a viscous potential flow analysis of capillary instability. Results of linearized analysis based on potential flow of a viscous and inviscid fluid were compared with the unapproximated normal mode analysis of the linearized Navier-Stokes equations. The growth rates for the inviscid fluid are largest, the growth rates of the fully viscous problems are smallest and those of viscous potential flow are between. The growth rates of the fully viscous fluid analysis and viscous potential flow are uniformly in good agreement. The results from all three theories converge when a Reynolds number $\gamma D\rho_l/\mu^2_l$ based on the velocity γ/μ_l of capillary collapse is large, (γ, D, ρ_l, μ_l) = (surface tension, diameter, density, viscosity). The convergence results apply to two liquids as well as to liquid and gas.

Funada and Joseph (2002b) did the same type of analysis of capillary instability of a viscoelastic fluid of the Maxwell model. The results are similar to those for viscous potential flow.

In a recent paper Joseph (2002) applied the theory of viscous potential flow to the problem of finding the rise velocity U of a spherical cap bubble (Davies and Taylor 1950, Batchelor 1967). The rise velocity is given by

$$\frac{U}{\sqrt{gD}} = -\frac{8}{3}\frac{\nu(1+8s)}{\sqrt{gD^3}} + \frac{\sqrt{2}}{3}\left[1 - 2s - \frac{16s\sigma}{\rho g D^2} + \frac{32\nu^2}{gD^3}(1+8s)^2\right]^{1/2} \tag{2}$$

where $R = D/2$ is the radius of the cap, ρ and ν are the density and kinematic viscosity of the liquid, σ is surface tension and $s = r''(0)/D$ is the deviation of the free surface

$$r(\theta) = R + ½\, r''(0)\theta^2 = R(1 + s\theta^2)$$

from perfect sphericity $r(\theta) = R$ near the stagnation point $s = 0$. The bubble nose is more pointed when $s < 0$ and blunted when $s > 0$. A more pointed bubble increases the rise velocity; the blunter bubble rises slower. The Davies-Taylor (1950) result $U = \frac{\sqrt{2}}{3}\sqrt{gD}$ arises when all other effects vanish; if s alone is zero,

$$\frac{U}{\sqrt{gD}} = -\frac{8}{3}\frac{\nu}{\sqrt{gD^3}} + \frac{\sqrt{2}}{3}\left[1 + \frac{32\nu^2}{gD^3}\right]^{1/2} \tag{3}$$

showing that viscosity slows the rise velocity. Equation (3) gives rise to a hyperbolic drag law

$$C_D = 6 + 32/R_e \tag{4}$$

which is in excellent agreement with the data on the rise velocity of spherical cap bubbles given by Bhaga and Weber 1981 (their figure 7, our figure 2).

C.1. (a) Topic 1: Extend the results of published works of quality based on inviscid potential flows to viscous and viscoelastic potential flows.

The entry “potential flow” in Google’s internet search engine gives rise to 2,230,000 hits. None of these, except possibly for the few works already cited, would be for viscous potential flow. Updating of even a tiny fraction of the papers on inviscid potential to viscous potential flow is at least a lifetime of work, even for a younger man.

The theory of two-dimensional potential flow based on complex variables is one of the most elegant examples of applied analysis. We propose to extend this theory to viscous potential flow. This is an extension of the classical theory in which the viscous stress is expressed in terms of second derivatives of the complex potential with respect to Z and $\overline{Z}$. This project may yield new theorems for this elegant theory because the processing of viscous stresses as a function of complex variables has not been done before.

We propose to continue our studies of interfacial dynamics, using viscous potential flow. At present, we are carrying out analysis of spatial, absolute and convective instability of liquid jets using viscous potential flow The analysis uses the method of Briggs (1964) and the singularity calculation of pinch mentioned on page 275 of the book by Schmid and Henningson (2001). If $D(k,\omega) = 0$ is the dispersion relation, then the singularities (pinch points) in the k plane must satisfy $\partial D(k,\omega)/\partial k = 0$ where k and ω are both complex valued. These singularities allow us to distinguish the conditions under which the flow is absolutely or convectively unstable. In the liquid jet case we get an explicit dispersion relation of the inviscid analysis $D(k,\omega) = 0$ and the generalization of the inviscid analysis to viscous potential flow gives then explicit results about the effects of viscosity. The results are gratifying since the criteria distinguishing absolute from convective stability is nearly the same as criteria derived by Lin and Lian (1989) who studied this problem for a viscous liquid jet without assuming potential flow. The same sort of explicit analysis based on viscous potential flow works perfectly well for two liquids, as well as for liquid and gas.

A bewildering number of excellent free surface problems solved by inviscid potential flow can be extended easily to include the effects of viscosity using viscous potential flow.

The entry "free surface flow" on Google gives rise to 294,000 hits with a large number of problems solved by inviscid potential flow such as, deformation of drops and bubbles in uniform flow, cavitation problems, jets rising and falling under gravity, jets ejected from square or elliptical orifices, weir flows, resonantly interacting water waves and many kinds of interface stability problems, and many other problems.

C.1. (b) Topic 2: Understand, characterize and evaluate the differences between dissipation approximations and viscous potential flow.

Viscous potential flow gives rise to better results than inviscid potential flow where "better" is relative to experiments or to solutions for which potential flow is not assumed. As a matter of principle vorticity will always be generated at an interface when the no-slip condition is applied. In many cases this vorticity is confined to a boundary layer whose effects are important in some cases or for some solutions properties and not important for others. A mathematical theory for the effects of vorticity layers and an understanding of when and where these effects are important are goals of this research.

Dissipation approximations are one way in which the effects of vorticity layers can be determined without actually calculating the layers. As far as I know there are only two cases in which the dissipation approximation has been applied; to the rise velocity of a spherical gas bubble and to the decay of gravity waves on water. Both cases present situations in which the effects of vorticity layers are important as well as those for which it is unimportant. The analysis of the relation of the dissipation approximation to viscous potential flow given below will point to the kind of discriminations which I wish to study in the proposed research.

The mechanical energy equation for the Navier-Stokes equations is

$$\frac{\rho}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_V \mathbf{u}^2 dV = \int_A \mathbf{u}\cdot(\mathbf{T}\cdot\mathbf{n})\,\mathrm{d}A - \int_V 2\mu\,\mathbf{D}:\mathbf{D}\,\mathrm{d}V \tag{5}$$

where

$$\mathbf{T} = -p\mathbf{1} + 2\mu\,\mathbf{D}[\mathbf{u}]$$

is the stress and $\mathbf{D}[\mathbf{u}]$ is the rate of strain tensor. For steady flow, the left-hand side of (5) vanishes. This equation was used by Levich (1949) to compute the rise velocity of a spherical gas bubble. He writes

$$DU = 2\mu\int_V \mathbf{D}:\mathbf{D}\,\mathrm{d}V = 2\mu\int_V \frac{\partial^2\phi}{\partial x_i \partial x_j}\frac{\partial^2\phi}{\partial x_i \partial x_j}\mathrm{d}V \tag{6}$$

where U is the rise velocity of the gas bubble and $D = \int \mathbf{e}_x \mathbf{T}\cdot\mathbf{n}\,\mathrm{d}A$ is the drag, A is the surface of the sphere and $\mathbf{u} = \nabla\phi$ where

$$\phi = -\frac{1}{2}Ua^3\frac{\cos\theta}{r^2} \tag{7}$$

is the potential for flow around a sphere. After putting (7) in (6), he finds that

$$D = 12\pi a\mu U, \qquad C_D = 48/R. \tag{8}$$

Equation (8) is the drag result of the dissipation approximation.

Analysis of the rising bubble based on viscous potential flow is the same as for inviscid potential flow except that the viscous contribution to the normal stress balance must be included. Moore (1959) applied the normal stress boundary condition to a spherical bubble using (7) and found

$$\mathbf{T}_{rr} = -p + 2\mu\frac{\partial u}{\partial r} = -p_I - 6\left(\frac{\mu U}{a}\right)\cos\theta \tag{9}$$

where p_I is the pressure from the potential flow solution; he put the tangential stress of the potential flow on the bubble surface as zero, and computed

$$D = 8\pi\mu Ua, \; C_D = 32/\mathrm{R} \tag{10}$$

Equation (10) may be considered to be the drag result of viscous potential flow.

To reconcile (8) and (10) in a later paper, Moore (1963) carried out a boundary layer analysis at the surface of the bubble and found that

$$C_D = \frac{48}{R}\left\{1 - \frac{2.2}{\sqrt{R}} + \cdots\right\} \tag{11}$$

The leading order agrees with the Levich formula (8). Kang and Leal (1988) obtained $48/R$ using another method to represent the effects of the vorticity layer (see figure 1). In this case, the rise of a small spherical bubble, it is necessary to compute the effects of the vorticity layer on viscous potential flow.

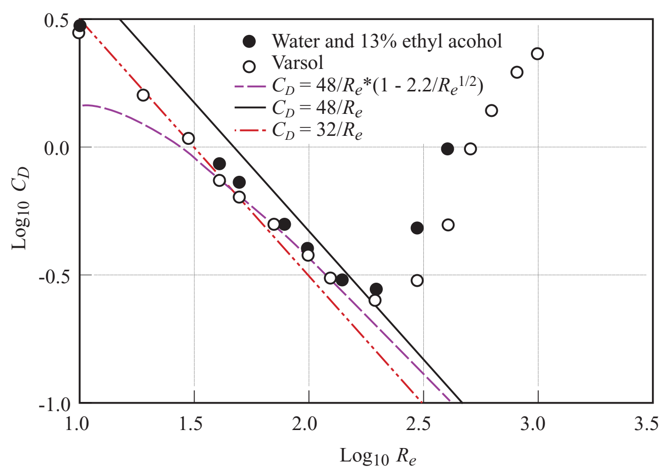

Figure 1. (after Batchelor 1967.) The drag coefficient of gas bubbles rising through liquids. The points for two particular liquids are taken from experimental curves given by Haberman and Morton (1953). The line $C_D = 32/R$ was added by me. The trend of the data may be more influenced by bubble deformation than by any effect of a vorticity boundary layer. It is hard to find 48/R in this data.

Large gas bubbles do not stay spherical; instead they take the lenticular shape of a spherical cap bubble. Davies and Taylor 1950 showed that the rise velocity of such a bubble could be obtained from a local analysis without using a drag balance, noting that the nose of the bubble is spherical as a result of the pressure generated by motion, without surface tension. Joseph 2002 generalized their inviscid potential flow result to include effects of viscosity, surface tension and the deviation of the bubble nose from sphericity using viscous potential flow and he obtained the formulas (2), (3) and (4) in this proposal.

Taylor and Davies 1950 result, and the drag formula (4) of viscous potential flow, are in excellent agreement with experiments reported by Bhaga and Weber 1981 after (4) is scaled so that the effective diameter used in the experiments and the spherical cap radius of Taylor are the same (see Figure 2).

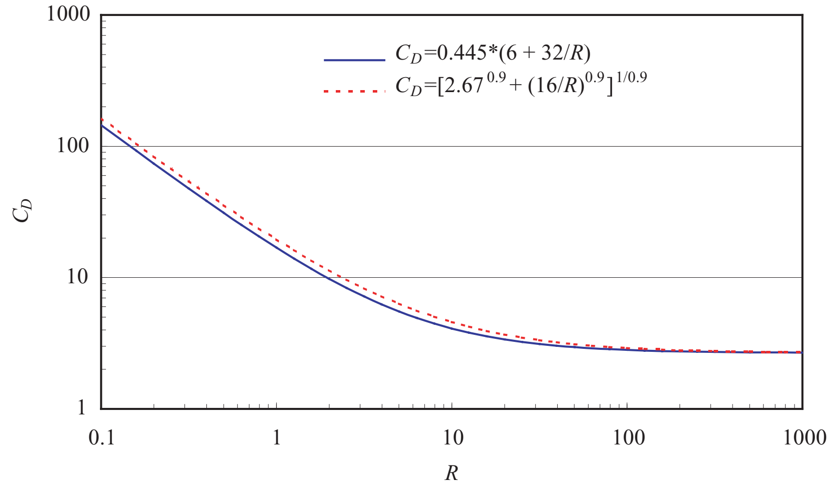

Figure 2. Comparison of the empirical drag law with the theoretical drag law (4) scaled by the factor 0.445 required to match the experimental data reported by Bhaga and Weber 1981 with the experiments of Davies and Taylor 1950 at large R_e.

Viscous potential flow appears to work well in the case of spherical cap bubbles, but a vorticity layer correction is required for small spherical gas bubbles. We need to understand this difference better than we do now.

We turn next to the dissipation approximation for the decay of water waves. Lamb (1924, p. 624) considered the effect of the viscous dissipation of a free traveling wave given by the potential

$$\phi = ace^{ky} \cos k\,(x\text{-}ct) \tag{12}$$

where c is the wave-velocity and $c = \sqrt{g/k}$ for inviscid potential flow. He found that the mean value of the dissipation per unit area is given by $2\mu k^3 a^2 c^2$. The kinetic energy per unit area is ¼ $\rho k a^2 c^2$, and the total energy (kinetic plus potential) is therefore double of this. Hence in the absence of surface forces $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\rho k c^2 a^2\right) = -2\mu k^3 a^2 c^2$, it follows that $\frac{\mathrm{d}\,a}{\mathrm{d}\,t} = -2\nu k^2 a$, or

$$a = a_0 e^{-2\nu k^2 t}\,. \tag{13}$$

Equation (13) gives the rate of decay of a free wave computed by dissipation approximation.

An analysis of the stability of gravity waves using viscous potential flow is embedded in the analysis of Kelvin-Helmholtz instability by Funada and Joseph (2001). A free wave is not stable, it must decay but at half the rate given by Lamb's dissipation calculation. In the analysis of linear stability of gravity waves based on viscous potential flow $\mathbf{u} = \nabla\phi$, $\nabla^2\phi = 0$ we find, after eliminating the pressure in the normal stress balance, that

$$\frac{\partial \phi}{\partial t} + g\eta + 2\nu \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{14}$$

and from the kinematic condition for the surface elevation $y = \eta$ we get

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} \tag{15}$$

on $y = 0$. After eliminating η in (14) using (15) and applying normal modes proportional to exp $k(y - ict)$ we find

$$c = -i\nu k \pm \sqrt{\frac{g}{k} - \nu^2 k^2} \tag{16}$$

Hence the normal solution is proportional to

$$e^{-\nu k^2 t} e^{ik\left(x \pm t\sqrt{\frac{g}{k} - \nu^2 k^2}\right)} \tag{17}$$

The amplitude of the wave decays at a rate

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\nu k^2 a, \tag{18}$$

one-half of the rate given by (13). The wave speed c is given by

$$c = \sqrt{\frac{g}{k} - \nu^2 k^2}, \qquad (19$$

which is slower than $\sqrt{g/k}$ for $k^3 < g/\nu^2$. For very large values of k, short standing waves do not propagate but simply decay at a rate given by

$$a = a_0 \exp\left(-\frac{1}{2}\frac{g}{\nu k} t\right). \tag{20}$$

Lamb (124, p. 625) also did direct calculation of the effect of viscosity on water-waves. He found 2D water-wave problem is satisfied by:

$$u = -\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}, \; v = -\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}. \tag{21}$$

and $\frac{p}{\rho} = \frac{\partial \phi}{\partial t} - gy$, provided $\nabla_1^2 \phi = 0$, $\frac{\partial \psi}{\partial t} = \nu \nabla_1^2 \psi$. (22)

where $\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. The solutions of (22) can be assumed to be

$$\phi = (Ae^{ky} + Be^{-ky})e^{ikx+nt}, \; \psi = (Ce^{my} + De^{-my})e^{ikx+nt}. \tag{23}$$

with $m^2 = k^2 + n/\nu$. It can be shown that B=0, D=0. Lamb gave the dispersion relationship:

$$(n + 2\nu k^2)^2 + gk + \gamma' k^3 = 4\nu^2 k^3 \sqrt{k^2 + n/\nu}. \tag{24}$$

where $\gamma' = \gamma / g$, γ is the surface tension coefficient.

When $\nu k^2 << \sqrt{gk + \gamma' k^3}$ (long waves),

$$n = -2\nu k^2 \pm i\sqrt{gk + \gamma' k^3}. \tag{25}$$

The decay rate agrees with the dissipation approximation result (13). When $\nu k^2 >> \sqrt{gk + \gamma' k^3}$ (short waves) and with γ' ignored,

$$n = -\frac{g}{2k\nu}. \tag{26}$$

The decay rate agrees with the viscous potential flow result (20).

As far as I know there are no measurements on the decay rate of gravity waves due to viscosity. The explanation of the differences between the dissipation approximation and viscous potential flow have yet to be given.

C.1. (c) Topic 3. Apply viscous potential flow to problems in which stresses computed on irrotational flow could be important.

Three such problems are proposed.

(i) Stress induced cavitation as the final stage of capillary collapse and rupture.

(ii) Stress induced cavitation in atomizers.

(iii) Stress induced cavitation due to ultrasound.

All of these problems look to effects of cavitation and are based on a theory of stress induced cavitation put forward by Joseph (1995, 1998). The theory is based on the observation that the pressure in a liquid is the mean normal stress and the liquid cannot average its stresses; the state of the stress at a point is relevant and the liquid will break or cavitate under tension. It is necessary to look at the state of stress in principal coordinates to compare the maximum tension with the breaking strength or cavitation threshold. A cavity will open in the direction of maximum tensile stress, which is 45° from the plane of shearing in pure shear of Newtonian fluid.

The computation of internal stresses due to motion should be considered in all problems in which cavitation is an issue. The calculation of principal stress at each point in a flow is greatly simplified when the velocity field is given by a potential and the stress by equation (1). This simplification has value for flows in which the regions of vorticity creation are confined to small layers near solid boundaries. Batchelor (1967, pg 398) said that results of irrotational flow "…may be applied directly to cases of flow at large Reynolds number in which boundary-layers separation does not occur (which would include slender bodies moving parallel to their length, and bodies of arbitrary shape accelerating from rest or executing translational or rotational oscillation of small amplitude about a fixed position…" Flows from reservoirs of fluid at rest in nozzles are irrotational near the nozzle entrance before the boundary layers build up. Batchelor (pg 276) notes that "… a body of inviscid fluid in irrotational continues to move irrotationally." I like better the statement on page 277 that, "The conditions under which irrotational motion remains irrotational are those for which Kelvin's circulation theorem is valid" because this theorem is valid for the irrotational flow of a viscous fluid. Potential flow simplifies the search for the relevant physics by simplifying the mathematics even in cases where more should be done.

C.1. (c) (i) Stress induced cavitation at the final stage of capillary collapse and rupture

Cavitation will occur in pure extension when the extensional stress is large enough, at high rates of extension. Lundgren and Joseph (1998[1]) looked this idea to elucidate the mechanism of rupture of a liquid cylinder under capillary collapse. They analyzed the breakup of a capillary filament as a viscous potential flow near a stagnation point on the centerline of the filament towards which the surface collapses under the action of surface tension forces. They found that the neck is of parabolic shape and its radius collapses to zero in a finite time. During the collapse the tensile stress due to viscosity increases in value until at a

[1] Available in PDF format from *http://www.aem.umn.edu/people/faculty/joseph/archive/docs/capillary.pdf.*

certain finite radius, which is about 1.5 microns for water in air, the stress in the throat passes into tension, presumably inducing cavitation there. The problem of capillary collapse or "pinching" has recently seen a burst of interest possibly due to the discovery of several similarity solutions (Eggers 1993, 1997; Papageogiou 1995) and others reviewed in the paper of McKinley and Tripathi (2000). These authors are not interested in the physics of rupture or breakup by cavitation and they do not compute stresses. All of the above mentioned authors find that capillary radius decreases to zero linearly in time, but the rate of collapse differs from author to author. McKinley and Tripathi (2000) write the formula

$$R_{mid}^{(t)} = R_1 - \frac{2X-1}{6}\frac{\gamma}{\eta}t \tag{27}$$

for the neck radius of the collapsing capillary in the stage of final decay as t increases to t^* when $R_{mid}^{(t^*)} = 0$. They give the X obtained by different authors in their table 1, but without the value $X = 2$ obtained by Lundgren and Joseph for viscous potential flow, who give the fastest decay. Eggers (1993, 1997) obtained $X = 0.5912$ and Papageoriou (1995) obtained $X = 0.719$. The solutions of the two authors last named have vorticity; Papageriou's solution has no inertia. Lundgren and Joseph found that the Reynolds number $\mathrm{Re} = R_{CR}\,\gamma/2\upsilon\eta$ based on velocity $v = \gamma/\eta$ of capillary collapse at the point of capillary collapse where $R_{CR} = 1$ micron is about 55.

Essential issues about the final collapse are suggested in the following citation of McKinley and Tripathi (2000):

> "Very close to the breakup event, Brenner [Lister and Stone] (1996) and Eggers (1997) note that the inertial effects can no longer be neglected in the fluid, since the local rate of extensional deformation $\left\lfloor \dot{\varepsilon} \approx \left(R_{mid}\right)^{-1} \partial R_{mid} / \partial t \right\rfloor$ diverges. In this region one should thus expect the solution given in Eq. (9) to cross over from the inertialess similarity solution with $X = 0.7127$, to the universal form discovered by Eggers with $X = 0.5912$…"

McKinley and Tripathi appear to believe that the final decay ought to be described by a similarity solution. However the idea of a cross-over implies that in the region of cross-over there is another non-similarity solution. Maybe the final decay does not go to Eggers solution.

In the state of final collapse, the extensional stress $\eta\partial u/\partial x$ leading to tension gets very large but the capillary pressure due to the decreasing radius of the jet leads to compression. We find that the sum of the two effects in the similarity solutions do not lead to tension but the viscous potential flow solution does lead to tension.

The solution of Lundgren and Joseph (1998) is local at the neck and it is not rigorous; it rests on several assumptions, like stagnation point flow in the neck that needs not be generated by a global viscous potential flow solution. The existence and properties of global potential solutions of the Navier-Stokes equations in the nonlinear case is an open question and the capillary collapse problem is just one realization. Viscous potential flow works well in the linear case and it ought to be studied for nonlinear problems.

We propose to apply viscous potential flow to the capillary collapse problem in the nonlinear case using numerical methods. We would look for capillary collapse on a periodic domain using a high resolution potential flow solver which would allow us to monitor the extensional stress at the final collapse, together with level set methods to resolve the interface conditions.

Studies of nonlinear problems based on viscous potential flow are necessary for the further evolution of this subject.

C.1. (c) (ii) Stress induced cavitation of liquids in atomizers

We propose to use potential flow to look at stress induced cavitation in supercavitating nozzles. Supercavitation is a name introduced by Knapp, Daily and Hammitt (1970) for geometry induced cavitation which collapses away from the object that initiated it. Chaves, Knapp, Kubitzek, Obermeier and Schneider (1995) note that "Above an injection pressure threshold that depends on the nozzle geometry and chamber pressure, cavitation appears at the sharp inlet corner of the nozzle. With increasing injection pressure the cavitation reaches the nozzle exit (supercavitation)." Reitz and Bracco (1982) identified four regimes of jet breakup; at the highest injection pressures the drop size is much smaller than the jet diameter with breakup observed already at the nozzle exit. Bergwerk (1959) had observed cavitation in nozzles, starting at the nozzle entrance. Reitz and Bracco (1982) proposed that cavitation in the nozzle might be a mechanism for atomization. The observations of Soteriou, Andrews and Smith (1995) as well as Chaves, *et al* (1995) are consistent with the hypothesis that the atomization is associated with supercavitation. Soteriou *et al* (1995) note that "The cavitating region consists of an opaque, creamy white foam…" which at one stage "…forms a ring close to the top of the hole."

The mechanism for supercavitation is not understood; it is believed to be associated with boundary layer separation.

We propose to explore the idea that cavitation at the exit ring is stress induced and to calculate the stress using potential flow for flow through an orifice under inlet conditions used for atomizers.

An interesting new approach to the problem of supercavitating nozzles is through the analysis of viscous potential flow through two-dimensional apertures. An elegant potential flow theory of flow through aperture based on complex variables can be found say in Lamb (1932). This theory can be used to calculate viscous stresses in the aperture. At each point in the aperture we can refer the stress to principal coordinates and identify the maximum tension there; the tension depends on viscosity and is different than the pressure. We expect cavitation at places where the tension exceeds the cavitation threshold. The calculation would compare points of minimum pressure with points of maximum tension.

C.1. (c) (iii) Stress induced cavitation due to ultrasound

First, I will do a simple calculation of ultrasound cavitation using the viscous stress (1) for potential flow, showing that viscosity should not be neglected. At this end, I will set up a more rigorous formulation for future work. There are many and diverse applications of cavitation induced by ultrasound. The entry "ultrasound cavitation" in Google's internet search engine gives rise to 3,970 hits; some of the entries on the first page are (1) ultrasound, cavitation and cleaning, (2) ultrasound cavitation pathways leading to suppression in tumor growth and enhancement in drug efficacy, (3) ultrasound – the key to better crystals, (4) ultrasound induced cavitation and sonochemical yields, and others.

Studies of ultrasound cavitation, and cavitation in general, are based on the idea that the liquid cavitates when the pressure passes through a cavitation threshold. The possible role of stress induced cavitation, using the idea that the liquid cavitates when the maximum tensile stress passes through a cavitation threshold, has not yet appeared in the literature.

A simple argument which demonstrates the importance of stress induced cavitation can be constructed from the most elementary study of one-dimensional acoustic wave propagation, based on potential flow, on pages 251-254 of Landau and Lifshitz (1987). In this calculation we neglect the dissipation term in the damped wave equation (50) and the dilitational contribution, proportional to div $\mathbf{u}$, in the stress (40).

The velocity potential is ϕ, $\boldsymbol{v}$ = grad ϕ and in the one-dimensional case there is one component of velocity $v = \partial\phi/\partial x$ where ϕ satisfies a wave equation (their (63.9)).

$$\frac{\partial^2\phi}{\partial x^2} - \frac{1}{C^2}\frac{\partial^2\phi}{\partial t^2} = 0 \tag{28}$$

where C is the speed of sound. The pressure p' and density ρ' perturb a uniform pressure and density p_0 and ρ_0. The perturbation pressure is related to the potential by

$$p' = -\rho_0\frac{\partial\phi}{\partial t} \tag{29}$$

and the stress is given by (1)

$$T_{xx} = -p_0 + \rho_0\frac{\partial\phi}{\partial t} + 2\mu\frac{\partial^2\phi}{\partial x^2} \tag{30}$$

where μ is the viscosity. Combining now (30) and (28) we find

$$T_{xx} = -p_0 + \rho_0\frac{\partial\phi}{\partial t} + \frac{2\mu}{C^2}\frac{\partial^2\phi}{\partial t^2} \tag{31}$$

The solution of the potential flow problem for monochromatic travelling plane waves is given by (64.19) of Landau and Lifshitz (1987) as

$$\phi = a\cos\left(\frac{\omega x}{C} - \omega t + \alpha\right) \tag{32}$$

where a is the amplitude of the wave and α is its phase. Using the expression (32) in equation (31), we find that

$$\begin{aligned} T_{xx} + p_0 = {} & a\rho_0\omega\sin\left(\frac{\omega x}{C} - \omega t + \alpha\right) \\ & - a\frac{2\mu\omega^2}{C^2}\cos\left(\frac{\omega x}{C} - \omega t + \alpha\right). \end{aligned} \tag{33}$$

Equation (33) shows that at each fixed point x, the extra stress $T_{xx} + p_0$ is an oscillating function whose amplitude increases with frequency.

The tension T_{xx} at x increases when the right hand side is positive. Assuming now that $a > 0$, without loss of generality, we find that at times t such that

$$\frac{\omega x}{C} - \omega t + \alpha = \frac{(4n+1)}{2}\pi \tag{34}$$

the extra stress

$$T_{xx} + p_0 = a\rho_0\omega \tag{35}$$

The tension T_{xx} here is increased by the perturbation pressure as in the classical case, with a zero contribution from the viscous extensional stress. On the other hand, at the times t such that

$$\frac{\omega x}{C} - \omega t + \alpha = (2n+1)\pi \tag{36}$$

the contribution to the tension

$$T_{xx} + p_0 = \frac{2a\mu\omega^2}{C^2} \tag{37}$$

is entirely due to the viscous stresses.

If we are allowed for the sake of argument to imagine that the linear theory being used here, in which a is indeterminate, applies to cavitation, we may look for the conditions under which

$$T_{xx} + p_0 = O(10^6) \text{ (dynes/cm}^2\text{)} \tag{38}$$

which puts T_{xx} near to a cavitation threshold conservatively estimated as $T_{xx} = 0$.

We next estimate the value of the extra stress, modulo the unknown amplitude a (cm^2/sec), for ultrasonic frequencies in water and glycerin with representative properties (see table 1) near room temperature.

	$\rho_0 \dfrac{\text{gm}}{\text{cm}^3}$	$\mu \dfrac{\text{gm}}{\text{cm sec}}$	$C \dfrac{\text{cm}}{\text{sec}}$	$\dfrac{\mu}{C^2} \dfrac{\text{gm}}{\text{cm}^3}$
Water	1	0.0114	1.48×10^5	0.52×10^{-12}
Glycerin	1.26	23.3	1.9×10^5	6.45×10^{-10}

Table 1. Properties of water and glycerin.

Two ultrasonic frequencies

$$\omega = 10^9 \text{ rad/sec } (f = 159.3\text{MHz}), \qquad \omega = 10^{10} \text{ rad/sec } (f = 1593\text{MHz}) \tag{39}$$

are selected for testing; the second frequency is in the high range.

Unfortunately the calculation of the tensile stress can be determined only up to an arbitrary multiplicative constant. The values of $T_{xx} + p_0$ given in table 2 do show that liquids are apt to cavitate at ultrasonic frequency, as is observed. The viscous contribution to cavitation cannot be neglected and it even becomes dominant at very high ultrasonic frequencies. The analysis suggests that the more viscous the fluid, the more apt it is to cavitate at high frequencies. The presence of the viscous component of T_{xx} means that high tensile stresses occur more frequently than they would if viscosity were neglected.

The analysis given here is heuristic but it suggests that ultrasound cavitation ought to take into account viscous stresses and these stresses may be considered within the frame of potential flow.

	$a\rho\omega$ dyne/cm^2		**$2a\mu\omega^2/c^2$ dyne/cm^2**	
	$\omega = 10^9$	**$\omega = 10^{10}$**	**$\omega = 10^9$**	**$\omega = 10^{10}$**
Water	$a\ 10^9$	$a\ 10^{10}$	$1.04a\ 10^6$	$1.04a\ 10^8$
Glycerin	$1.26a\ 10^9$	$1.26a\ 10^{10}$	$1.29a\ 10^9$	$1.29a\ 10^{11}$

Table 2. Calculation of $T_{xx} + p_0$ from (35) and (37) using ultrasonic frequencies (39). The viscous contribution to the maximum tensile stress cannot be neglected in cavitation due to ultrasound at high frequencies.

The theory of viscous potential flow has not yet been applied to the compressible Navier-Stokes equations. In general, vorticity is generated by gradients of ρ and μ and the curl $\mathbf{u} = 0$ is not a solution of the compressible equations. The stress for a compressible viscous fluid is given by

$$T_{ij} = -\left(p + \frac{2}{3}\mu \operatorname{div}\mathbf{u}\right)\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{40}$$

Here, the second coefficient of viscosity is selected so that $T_{ii} = -3p$. (The results to follow will apply also to the case when other choices are made for the second coefficient of viscosity.) The equations of motion are given by $\rho\, d\mathbf{u}/dt = \operatorname{div} \mathbf{T}$ together with $d\rho/dt + \rho \operatorname{div} \mathbf{u} = 0$.

To study acoustic propagation, the equations are linearized; putting

$$[\mathbf{u}, p, \rho] = [\mathbf{u}', p_0 + p', \rho_0 + \rho'] \tag{41}$$

where $\mathbf{u}'$, p' and ρ' are small quantities, we get

$$T_{ij} = -\left(p_0 + p' + \frac{2}{3}\mu_0 \operatorname{div}\mathbf{u}'\right)\delta_{ij} + \mu_0\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right) \tag{42}$$

$$\rho_0 \frac{\partial \mathbf{u}'}{\partial t} = -\nabla p' + \mu_0\left(\nabla^2\mathbf{u}' + \frac{1}{3}\nabla \operatorname{div}\mathbf{u}'\right) \tag{43}$$

$$\frac{\partial \rho'}{\partial t} + \rho_0 \operatorname{div}\mathbf{u}' = 0 \tag{44}$$

where p_0, ρ_0 and μ_0 are constants. For acoustic problems, we assume that a small change in ρ induces small changes in p by fast adiabatic processes; hence

$$p' = C_0^2 \rho' \tag{45}$$

where C_0 is the speed of sound. From (43) we find that curl $\mathbf{u}' = 0$ is a solution of the vorticity equation and we may introduce a potential

$$\mathbf{u}' = \nabla\phi. \tag{46}$$

Combining next (46) and (43), we get

$$\nabla\left[\rho_0 \frac{\partial\phi}{\partial t} + p' - \frac{4}{3}\mu_0 \nabla^2\phi\right] = 0. \tag{47}$$

The quantity in the bracket is equal to an arbitrary function of the time which may be absorbed in ϕ.

A viscosity dependent Bernoulli equation

$$\rho_0 \frac{\partial\phi}{\partial t} + p' - \frac{4}{3}\mu_0 \nabla^2\phi = 0 \tag{48}$$

is implied by (47). The stress (42) is given in terms of the potential ϕ by

$$T_{ij} = -\left(p_0 - \rho_0 \frac{\partial\phi}{\partial t} + 2\mu_0 \nabla^2\phi\right)\delta_{ij} + 2\mu_0 \frac{\partial^2\phi}{\partial x_i \partial x_j}. \tag{49}$$

To obtain the equation satisfied by the potential ϕ, we eliminate ρ' in (44) with p' using (45), then eliminate $\mathbf{u}' = \nabla\phi$ and p' in terms of ϕ using (48) to find

$$\frac{\partial^2\phi}{\partial t^2} = \left(C_0^2 + \frac{4}{3}\nu_0 \frac{\partial}{\partial t}\right)\nabla^2\phi \tag{50}$$

where the potential ϕ depends on the speed of sound and the kinematic viscosity $\nu_0 = \mu_0/\rho_0$.

The damped wave equation (50) may be derived directly without introducing a potential from the compressible Navier-Stokes equation in the acoustic approximation; obviously, the viscosity dependent

Bernoulli equation (48) requires one to introduce a potential. Lamb (1932) derived (50) for the velocity in one space dimension directly from the compressible linearized Navier-Stokes equation for plane waves in a laterally unbounded medium (his equation (4), page 647). Lighthill (1978) derived the same one-dimensional damped wave equation for the density rather than the velocity without introducing a velocity potential. In Lighthill's equation (205) on page 79, ${}^{4}/_{3}\,\nu_0$ is replaced by δ, a relaxation time for a relaxing gas given by his equation (200), which may be written as

$$p' = C_0^2 \rho' + \delta \frac{\partial \rho'}{\partial t} \; . \tag{51}$$

Inserting (51) into (48) we get

$$\rho_0 \frac{\partial \phi}{\partial t} + C_0^2 \rho' + \delta \frac{\partial \rho'}{\partial t} - \frac{4}{3} \mu_0 \nabla^2 \phi = 0 \, . \tag{52}$$

Combining now (52) with

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla^2 \phi = 0 \, ,$$

we find a generalized damped wave equation

$$\frac{\partial^2 \phi}{\partial t^2} = \left(C_0^2 + \left[\delta + \frac{4}{3} \nu_0 \right] \frac{\partial}{\partial t} \right) \nabla^2 \phi \, . \tag{53}$$

All of the potential flow solutions which perturb the state of rest of an inviscid compressible fluid can be considered for the effects of viscosity using the potential flow equations for viscous compressible flows derived here. Under ordinary circumstances viscous and relaxation effects will be negligible. However, in problems of high frequency ultrasound, especially in problems of ultrasound cavitation of liquids, these effects are important, even dominant. The viscous effects which would enter into the study of stress induced cavitation due to high frequency ultrasound are dissipative effects more or less described by a telegraph equation and "anisotropic" pressure associated with the viscous part of the stress tensor (49).

C. 2. Results from prior NSF support

This is a regular renewal proposal. The NSF grant which we propose to renew is

(a) NSF/CTS 0076648, 8/31/01—8/31/03, $262,133.

(b) Comparative Theoretical and Experimental Studies of Breakup, Outgassing and Stress Induced Cavitation of Newtonian and Polymerically Thickened Liquids

(c) Summary of the Results of completed work

(i) Project Goals

In the prior grant we proposed studies of drop breakup due to flow-introduced stresses induced by instability; Rayleigh-Taylor, Kelvin-Helmholtz and capillary instability. We proposed studies of stress-induced cavitation in the flow puts the liquid into tension and breakup due to outgassing of dissolved gases. We proposed to prove that polymer additives increase the solubility in thickened liquids.

(ii) Findings

We solved all the above mentioned stability problems using viscous potential flow. We proved that viscous potential flow gives rise to accurate results compared to exact analysis and always to much better results than inviscid potential flow. We think that all the thousands of papers done on inviscid potential flow should and will be done doing viscous potential flow. We developed a theory of outgassing of dissolved gases under rapid depressurization in foamy oils in porous media. We found that outgassing in polymerically thickened liquids is due to water vapor and that polymer additives do not increase the solubility of liquids.

The flow of foamy oil in porous media is a problem for oil production in reservoirs of heavy oils in Venezuela and Canada. We developed a nucleation model of foamy oil flow which explains why the production and rate of production in these reservoirs under depressurization is anomously high. The theory of stress induced cavitation is a revision of the classical theory. These works impact petroleum engineering and chemical engineering.

(iii) Contribution to education and human resources

We employed large numbers of undergraduates in this project. These were upper end students and they were all enthusiastic about the opportunity to learn real research hands-on. Four women were employed on this project. At present, four graduate students and one woman post doc work on Joseph's NSF projects.

(d) Publications resulting from the NSF support

The following papers can be found at the web site *http://www.aem.umn.edu/people/faculty/joseph/ViscousPotentialFlow/*.

2002

Joseph, D.D., J. Wang, 2002. Potential flow solutions of the compressible Navier-Stokes equation in the acoustic approximation, under review.

Joseph, D.D., 2002. Rise velocity of spherical cap bubble, under review.

Funada, T., D.D. Joseph, 2002. Viscous potential flow analysis of the spatial absolute and convective instability of a liquid jet. *J. Fluid Mech.*, submitted.

Funada, T. and D.D. Joseph, 2002. Viscoelastic potential flow analysis of capillary instability, *J. Non-Newtonian Fluids*, submitted.

Joseph, D.D., A.M Kamp, T. Ko, R. Bai, 2002. Modeling Foamy Oil Flow in Porous Media II: Nonlinear Relaxation Time Model of Nucleation, *Int. J. Multiphase Flow,* submitted.

Joseph, D.D., A.M. Kamp, R. Bai, 2002. Modeling Foamy Oil Flow in Porous Media, *Int. J. Multiphase Flow*, **28**(10), 1659-1686.

Funada, T., D.D. Joseph, 2002. Viscous potential flow analysis of capillary instability*, Int. J. Multiphase Flow,* **28**(9), 1459-1478.

Joseph, D.D., G.S. Beavers, T. Funada, 2002. Rayleigh-Taylor instability of viscoelastic drops at high Weber numbers, *J. Fluid Mech*., **453**, 109-132.

2001

Funada, T., D.D. Joseph, 2001. Viscous potential flow analysis of Kelvin-Helmholtz instability in a channel, *J. Fluid Mech.,* **445**, 263-283.

Pereia, A., G. McGrath, D.D. Joseph, 2001. Flow and stress induced cavitation in a journal bearing with axial throughput, *J. Tribology*, **123**, 742-754.

2000 and prior

Joseph, D.D., J. Belanger, G.S. Beavers, 1999. Breakup of a liquid drop suddenly exposed to a high-speed airstream, *Int. J. Multiphase Flow*, **25**, 1263-1303.

Joseph, D.D. and T.Y. Liao, 1994a. Potential flows of viscous and viscoelastic fluids. *J. Fluid Mech*., **265**, 1-23.

Joseph, D.D. and T.Y. Liao, 1994b. Viscous and viscoelastic potential flow, *Trends and Perspectives in Applied Mathematics, Applied Mathematical Sciences,* Sirovich, Arnol'd, eds, Springer-Verlag, **100**, 1-54.

(e) Data and research not described elsewhere

Our database on drop breakup is being used by the Army for prediction of breakup of dangerous agents. We established a web page for these results where movies can be seen.

http://www.aem.umn.edu/research/Aerodynamic_Breakup/

(f) Relation of completed work to proposed work

We have extracted the most innovative parts of the previous research, viscous and viscoelastic potential flow and stress induced cavitation for further study. The proposed study is theoretical and mathematical and at present has no experimental component.