Interfacial shapes in the steady flow of a highly viscous dispersed phase

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Here we develop a perturbation theory for the steady flow of immiscible liquids when the dispersed phase is much more viscous than the continuous phase, as is the case in emulsions of highly viscous bitumen in water and in water lubricated pipelines of heavy crude. The perturbation is non-singular, but non-standard; the partitioning of the boundary conditions at different orders is not conventional. At zeroth order the dispersed phase moves as a rigid solid with an as yet unknown, to-be-determined, pressure. The flow of the continuous phase at zeroth order is determined by a Dirichlet problem with prescribed velocities on a to-be-iterated interfacial boundary. The first order problem in the dispersed phase is determined from the solution of a Stokes flow problem driven by the previously determined shear strain on the as yet undetermined interfacial boundary. This Stokes problem determines the unknown, to-be-determined, lowest order pressure distribution. At this point we have enough information to test the balance of normal stresses at lowest order; by iterating the interface shapes we may now complete the description of the lowest order problems.

The perturbation sequence in powers of the viscosity ratio has a similar structure at every order and all the problems may be solved sequentially with the caveat that the interface shape must be determined iteratively in each perturbation loop. Problems in which the internal motions of the dispersed phase are slow and slowly varying can also be treated with the same perturbation scheme.

1. Introduction

In treating the flow of two immiscible liquids with greatly different viscosities, like bitumen and water, certain simplifications arise when the more viscous liquid is dispersed and not attached to rigid boundaries. In this case the dispersed phase may move nearly as a rigid body since the forces which arise from the motion of the continuous phase are not great enough to drive large secondary motions in the dispersed phase. The water will move bitumen dispersed in water more or less as a rigid body provided that the bitumen is not anchored at some wall.

Here and henceforward we shall call the dispersed phase oil and the continuous phase water. We search for simplified mathematical descriptions as a perturbation of a rigid motion in the limit in which the ratio of the water viscosity μ_w to the oil viscosity μ_o

$$\varepsilon = \frac{\mu_w}{\mu_o} \to 0 \tag{1.1}$$

In this paper we will confine our attention to the cases in which interfacial rheology, and Maragnoni effects are neglected. These effects are greatly diminished by the high bulk viscosity of the dispersed phase and in a later work we will look to describe exactly how diminished these effects are. Generally speaking, our work here is motivated by the needs of the heavy oil industry.

2. Governing Equations

To keep the description simple, we consider the case when the oil is free to move in the water as in the case of sedimentation of a single drop of heavier-than-water oil or in the core annular flow studied by Bai, Kelkar and Joseph [1996].

In steady flow the oil-water interface is given by

$$F(\boldsymbol{\chi}(\varepsilon), \varepsilon) = 0 \tag{2.1}$$

where $\mathbf{x} = \boldsymbol{\chi}(\varepsilon)$ is the position of points on $F = 0$. The unknowns in our problem are

$$\left.\begin{array}{ll} \mathbf{u}(\boldsymbol{\chi}, \varepsilon), \psi(\mathbf{x}, \varepsilon) & \text{in the oil} \\[2mm] \boldsymbol{v}(\mathbf{x}, \varepsilon), \phi(\mathbf{x}, \varepsilon) & \text{in the water} \\[2mm] \boldsymbol{\chi}(\varepsilon) & \end{array}\right\} \tag{2.2}$$

where $\mathbf{u}$ and $\boldsymbol{v}$ are velocities and

$$\left.\begin{array}{l} \psi = p_o + \rho_o \boldsymbol{\chi} \cdot \mathbf{x} \\[2mm] \phi = p_w + \rho_w \boldsymbol{\lambda} \cdot \mathbf{x} \end{array}\right\} \tag{2.3}$$

are "dynamic" pressures, p is pressure and $\boldsymbol{\lambda}$ is a constant vector ($\boldsymbol{\lambda} = \mathbf{g}$ in sedimentation problems; $\boldsymbol{\lambda} = \mathbf{e}_x \beta$ for the constant part of the pressure gradient which balances the pressure drop in core-annular flow).

The equations of motion in the oil and water are

$$\rho_o \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \psi + \frac{\mu_w}{\varepsilon} \nabla^2 \mathbf{u}, \operatorname{div} \mathbf{u} = 0, \tag{2.4}$$

and

$$\rho_w \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla \phi + \mu_w \nabla^2 \boldsymbol{v}, \operatorname{div} \boldsymbol{v} = 0. \tag{2.5}$$

At the interface, the velocity is continuous

$$\mathbf{u}(\boldsymbol{\chi}) = \boldsymbol{v}(\boldsymbol{\chi}) \tag{2.6}$$

and the normal component vanishes

$$\mathbf{u}(\boldsymbol{\chi}) \cdot \mathbf{n} = \boldsymbol{v}(\boldsymbol{\chi}) \cdot \mathbf{n} = 0 \tag{2.7}$$

where $\mathbf{n}$ is the normal from oil to water. The shear stress is continuous

$$\boldsymbol{\tau} \cdot \mathbf{D}[\mathbf{u}(\boldsymbol{\chi}) - \varepsilon \boldsymbol{v}(\boldsymbol{\chi})] \cdot \mathbf{n} \tag{2.8}$$

where $\mathbf{D}[\mathbf{u}]$, the rate of strain, is the symmetric part of $\nabla \mathbf{u}$ and $\boldsymbol{\tau}$ is a unit tangent vector in the interface, $\boldsymbol{\tau} \cdot \mathbf{n} = 0$. The balance of normal stresses can be expressed as

$$-\phi(\boldsymbol{\chi}) + \psi(\boldsymbol{\chi}) - (\rho_w - \rho_o)\boldsymbol{\lambda} \cdot \boldsymbol{\chi}$$
$$+2\mu_w \mathbf{n} \cdot \mathbf{D}[\boldsymbol{v} - \mathbf{u}/\varepsilon] \cdot \mathbf{n} = 2H(\boldsymbol{\chi})\sigma \tag{2.9}$$

where $H(\boldsymbol{\chi})$ is the mean curvature and σ is interfacial tension.

The boundary conditions apply only to water, since oil is assumed not to touch the boundary. For steady flow the velocity of the bondary at $\mathbf{x} = \mathbf{x}_b$ is

$$v(\mathbf{x}_b) = \mathbf{V}. \tag{2.10}$$

$\mathbf{V}$ is the velocity of solid walls in a coordinate system centered on the falling drop or in a coordinate system moving with the average velocity of the core in annular flow.

3. Equations when $\varepsilon \to 0$

Assuming now that all fuctions listed in (2.3) are bounded as $\varepsilon \to 0$, we find that

$$\left. \begin{array}{l} \mathbf{u}_o(0) = 0, \\[2mm] \operatorname{div} \mathbf{u}_o = 0, \\[2mm] \nabla^2 \mathbf{u}_o = 0, \\[2mm] \mathbf{u}_o(\boldsymbol{\chi}_o) \cdot \mathbf{n}_o = 0, \\[2mm] \boldsymbol{\tau}_o \cdot \mathbf{D}[\mathbf{u}_o(\boldsymbol{\chi}_o)] \cdot \mathbf{n}_o, \\[2mm] \mathbf{n}_o \cdot \mathbf{D}[\mathbf{u}_o(\boldsymbol{\chi}_o)] \cdot \mathbf{n}_o. \end{array} \right\} \tag{3.1}$$

The function

$$\mathbf{u}_o(\mathbf{x}) \equiv 0$$

satisfies (2.10). Then, in the water we have

$$\left. \begin{array}{l} \rho_w \boldsymbol{v}_o \cdot \nabla \boldsymbol{v}_o = -\nabla \phi_o + \mu_w \nabla^2 \boldsymbol{v}_o, \operatorname{div} \boldsymbol{v}_o = 0 \\[2mm] v_o(\boldsymbol{\chi}_o) = 0 \\[2mm] v(\boldsymbol{\chi}_b) = \mathbf{V} \end{array} \right\} \tag{3.2}$$

Equations (3.2) are a Dirichlet problem for $\boldsymbol{v}_o(\mathbf{x})$ and $\phi_o(\mathbf{x})$ which can be solved when the interface $\boldsymbol{\chi}_o$ is given. No condition on $\boldsymbol{v}(\mathbf{x})$ arises from the shear stress balance (2.8); and shear stress arising from (3.1) is acceptable. The idea is to iterate $\boldsymbol{\chi}_o$, using the $\boldsymbol{\chi}_o$ that will reduce (2.9) to an identiy. To do this iteration, more work is required.

4. Perturbation equations at lowest order

Now we develop a solution in powers of ε, to the lowest order

$$
\left.\begin{aligned}
\mathbf{u}(\mathbf{x}, \varepsilon) &= \varepsilon \mathbf{u}_1(\mathbf{x}), \\
\psi(\mathbf{x}, \varepsilon) &= \psi_o(\mathbf{x}) + \varepsilon \psi_1(\mathbf{x}), \\
\upsilon(\mathbf{x}, \varepsilon) &= \upsilon_o(\mathbf{x}) + \varepsilon \boldsymbol{\upsilon}_1(\mathbf{x}), \\
\phi(\mathbf{x}, \varepsilon) &= \phi_o(\mathbf{x}) + \varepsilon \phi_1(\mathbf{x}), \\
\boldsymbol{\chi}(\varepsilon) &= \boldsymbol{\chi}_o + \varepsilon \boldsymbol{\chi}_1.
\end{aligned}\right\}
\tag{4.1}
$$

At the interface, we have

$$
\boldsymbol{\upsilon}(\boldsymbol{\chi}(\varepsilon), \varepsilon) = \boldsymbol{\upsilon}_o(\boldsymbol{\chi}_o) + \varepsilon \boldsymbol{\upsilon}_1(\boldsymbol{\chi}_o) + \varepsilon \psi_1 \cdot \nabla \boldsymbol{\upsilon}_o(\boldsymbol{\chi}_o),
\tag{4.2}
$$

$$
\phi(\boldsymbol{\chi}(\varepsilon), \varepsilon) = \phi_o(\boldsymbol{\chi}_o) + \varepsilon \phi_1(\boldsymbol{\chi}_o) + \varepsilon \boldsymbol{\chi}_1 \cdot \nabla \phi_o(\boldsymbol{\chi}_o)
\tag{4.3}
$$

Since $\mathbf{u}_o(\mathbf{x}) \equiv 0$ in the oil

$$
\mathbf{u}(\boldsymbol{\chi}(\varepsilon), \varepsilon) = \varepsilon \mathbf{u}_1(\boldsymbol{\chi}_o) + \varepsilon^2 \mathbf{u}_2(\boldsymbol{\chi}_o) + \varepsilon^2 \boldsymbol{\chi}_1 \cdot \nabla \mathbf{u}_1(\boldsymbol{\chi}_o)
\tag{4.4}
$$

but

$$
\psi(\boldsymbol{\chi}(\varepsilon), \varepsilon) = \psi_o(\boldsymbol{\chi}_o) + \varepsilon \psi_1(\boldsymbol{\chi}_o) + \varepsilon \boldsymbol{\chi}_1 \cdot \nabla \psi_o(\mathbf{x}_o).
\tag{4.5}
$$

Moreover, since the shape of drop changes with

$$
\begin{aligned}
\mathbf{n}(\boldsymbol{\chi}) &= \mathbf{n}_o + \varepsilon \mathbf{n}_1 \\
\boldsymbol{\tau}(\boldsymbol{\chi}) &= \boldsymbol{\tau}_o + \varepsilon \boldsymbol{\tau}_1
\end{aligned}
\tag{4.6}
$$

After inserting (4.1) through (4.6) into the basic equations (2.4) through (2.9) we find first that

$$
\left.\begin{aligned}
\nabla \psi_o &= \mu_w \nabla^2 \mathbf{u}_1, \; \operatorname{div} \mathbf{u}_1 = 0, \\
\mathbf{u}_1(\boldsymbol{\chi}_o) \cdot \mathbf{n}_o &= 0, \\
\boldsymbol{\tau}_o \cdot \mathbf{D}[\mathbf{u}_1(\boldsymbol{\chi}_o) - \boldsymbol{\upsilon}_o(\chi_o)] \cdot \mathbf{n}_o &= 0
\end{aligned}\right\}
\tag{4.7}
$$

This problem may be solved for $\mathbf{u}_1(x)$, and $\psi_o(\mathbf{x})$ when $\boldsymbol{\chi}_o$ is given. The slow motion in the oil is driven by

the shear rate in the water

$$\boldsymbol{\tau}_o \cdot \mathbf{D}[\boldsymbol{v}_o] \cdot \mathbf{n}_o = \partial v_\tau(\boldsymbol{\chi}_o)/\partial y_n \stackrel{\text{def}}{=} \overset{\circ}{\gamma}(\boldsymbol{\chi}_o) \tag{4.8}$$

where $v_\tau(\boldsymbol{\chi}_o)$ is the velocity component tangent to the interface and y_n is normal at $\mathbf{x} = \boldsymbol{\chi}_o$.

The normal stress balance (2.9) now becomes

$$-\phi_1(\boldsymbol{\chi}_o) + \psi_o(\boldsymbol{\chi}_o) - (\rho_w - \rho_o)\mathbf{g} \cdot \boldsymbol{\chi}_o$$
$$-2\mu_w \mathbf{n}_o \cdot \mathbf{D}[\mathbf{u}_1(\boldsymbol{\chi}_o)] \cdot \mathbf{n}_o = 2H(\boldsymbol{\chi}_o)\sigma \tag{4.9}$$

We may write

$$\mathbf{n}_o \cdot \mathbf{D}[\mathbf{u}_1(\boldsymbol{\chi}_o)] \cdot \mathbf{n}_o = \partial u_{1n}/\partial y_n$$

where u_{1n} is the normal component of $\mathbf{u}_1$ at the interface point $\mathbf{x} = \boldsymbol{\chi}_o$. In deriving (4.9) we used an easily proved result which says that

$$\mathbf{n}_o \cdot \mathbf{D}[\boldsymbol{v}(\boldsymbol{\chi}_o)] \cdot \mathbf{n}_o = 0$$

when $\boldsymbol{v}_o(\chi_o)$ is the fluid velocity at the boundary of a rigid body. Equation (4.9) selects $\boldsymbol{\chi}_o$ which until now was arbitrary.

5. Perturbation Equations at Higher Order

Continuing now to higher orders, we find that

$$\rho_w[\boldsymbol{v}_o \cdot \nabla \boldsymbol{v}_1 + \boldsymbol{v}_1 \cdot \nabla \boldsymbol{v}_o] = -\nabla \phi_1 + \mu_w \nabla^2 \boldsymbol{v}_1, \text{div } \boldsymbol{v}_1 = 0 \tag{5.1}$$

and

$$\boldsymbol{v}_1(\boldsymbol{\chi}_o) = \mathbf{u}_1(\boldsymbol{\chi}_o) - \boldsymbol{\chi}_1 \cdot \nabla \boldsymbol{v}_o(\boldsymbol{\chi}_o), \boldsymbol{v}_1(\boldsymbol{\chi}_b) = 0 \tag{5.2}$$

Equations (5.1) and (5.2) may be solved for $\boldsymbol{v}_1(\mathbf{x})$ and $\phi_1(\mathbf{x})$ when $\mathbf{x}_1$ is given.

To get $\boldsymbol{\chi}_1$, we must go to order ε^2 in our expansion. From (2.4) we get

$$0 = -\nabla \psi_1 + \mu_w \nabla^2 \mathbf{u}_2, \text{ div } \mathbf{u}_2 = 0 \tag{5.3}$$

Equation $(2.7)_1$ gives rise to

$$\mathbf{n}_o \cdot (\mathbf{u}_2(\boldsymbol{\chi}_o) + \boldsymbol{\chi}_1 \cdot \nabla \mathbf{u}_1(\boldsymbol{\chi}_o)) + \mathbf{n}_1 \cdot \mathbf{u}_1(\boldsymbol{\chi}_o) = 0. \tag{5.4}$$

and (2.8) leads to

$$\boldsymbol{\tau}_o \cdot \mathbf{D}[\mathbf{u}_2 - \boldsymbol{v}_1 + \boldsymbol{\chi}_1 \cdot \nabla(\mathbf{u}_1 - \boldsymbol{v}_o)] \cdot \mathbf{n}_o$$

$$+\boldsymbol{\tau}_1 \cdot \mathbf{D}[\mathbf{u}_1 - \boldsymbol{v}_o] \cdot \mathbf{n}_o + \boldsymbol{\tau}_o \mathbf{D}[\mathbf{u}_1 - \boldsymbol{v}_o] \cdot \mathbf{n}_1 = 0 \tag{5.5}$$

Equations (5.3), (5.4) and (5.5) can be solved for $\mathbf{u}_2$ and $\boldsymbol{\psi}_1$ when $\boldsymbol{\chi}_1$ is given. The normal stress condition (2.9) at order ε^2 gives rise to

$$-\phi_1(\boldsymbol{\chi}_o) + \psi_1(\boldsymbol{\chi}_o) - \boldsymbol{\chi}_1 \cdot \nabla(\phi_o - \psi_o)$$

$$-(\rho_w - \rho_o)\tfrac{\lambda}{h} \cdot \boldsymbol{\chi}_1 + 2\mu_w \mathbf{n}_o \mathbf{D}[\boldsymbol{v}_1 - \mathbf{u}_2 + \psi_1 \cdot \nabla(\boldsymbol{v}_o - \mathbf{u}_1)] \cdot \mathbf{n}_o \tag{5.6}$$

$$2\tfrac{dH}{d\boldsymbol{\chi}}(\boldsymbol{\chi}_o) \cdot \boldsymbol{\chi}_1 \sigma$$

Equation (5.6) selects the correct boundary perturbation.

Equations (2.5) and (2.6) at order ε^2 give rise to a perturbation problem for $\boldsymbol{v}_2$ and ϕ_2, depending on $\boldsymbol{\chi}_2$ and so on

$$\boldsymbol{\chi} = \mathbf{e}_x x + \mathbf{e}_z \zeta(x, \varepsilon)$$

$$\boldsymbol{\chi} = \mathbf{e}_z \zeta_1(x), \zeta_1 = \tfrac{\partial \zeta}{\partial \varepsilon}$$

So $\boldsymbol{\chi}_1$ is just one scalar function.

6. Core annular flow

Here we shall revisit the problem of waves on core-annular flow considered by Bai, Kelkar and Joseph [1997]. They treated a steady flow in which the holdup ratio c_o/c_w of average velocities $c_o = Q_o/\pi R_1^2$ and $c_w = Q_w/\pi(R_2^2 - R_1^2)$ is prescribed. Here Q_o and Q_w are the volume flux of oil and water, R_2 is the outer radius of the pipe and R_1 is the mean radius of the core. In the approximation carried out by them, the core is rigid. The analysis of the steady flow of water is carried out in a coordinate system in which the core is stationary; secondary motions in the core were not treated. The shape of the interface was computed using the normal stress condition under the assumption that the pressure in the core is uniform apart from a constant pressure gradient β along the pipe axis z (see figure 1).

[Figure 1 goes near here]

The problem of core annular flow may be treated in the framework of the perturbation theory described in section 3 and 4 with $\boldsymbol{\lambda} \cdot \mathbf{x}$ in (2.3) equal to $-\beta z$ where β is a constant pressure gradient. The governing equations at zeroth order are essentially (3.2).

$$\rho_w \boldsymbol{v} \cdot \nabla \boldsymbol{v} = \beta \mathbf{e}_z - \nabla p_w + \mu_w \nabla^2 = 0,$$

$$\boldsymbol{v} = 0 \text{ on } r = f(z), \tag{6.1}$$

$$\boldsymbol{v} = -c \mathbf{e}_z \text{ on } r = R_2$$

where $r = f(z)$ gives the shape of the interface and $f(z)$ was determined by Bai et al [1996] using the normal stress condition

$$\frac{\sigma}{f(1+f'^2)^{\frac{1}{2}}} - \frac{\sigma f''}{(1+f'^2)^{\frac{3}{2}}} = C - p_w. \tag{6.2}$$

The ratio of the average oil to water velocity $h = c/c_w$ is given by

$$h = \frac{Q_o/Q_w}{R_1^2/(R_2^2 - R_1^2)} = \frac{\pi c R_1^2}{\pi c [f^2 - R_1^2] + 2\pi \int_f^{R_2} r v dr} \frac{R^2 - R_1^2}{R_1^2} \tag{6.3}$$

Though f depends on z, h is a constant, independent of z; $h = 2$ for perfect core flow without waves and $h = 1$ when the water is trapped between wavecrests touching the pipe wall. For wavy flow $1 < h < 2$; $h = 1.4$ occurs frequently in experiments; the selection mechanism is related to stability and is not understood. Bai et al [1996] prescribed $h = 1.4$, ensuring waves.

Going further now than Bai et al [1997] we consider now the problem (4.7) for the flow $\mathbf{u} = \mathbf{u}_1$ in the oil core

$$-\beta \mathbf{e}_z + \nabla p_o = \mu_w \nabla^2 \mathbf{u}, \operatorname{div}\mathbf{u} = 0 \tag{6.4}$$

where, on $r = f(z)$, we have

$$\mathbf{u}(r, z) \cdot \mathbf{n} = 0 \tag{6.5}$$

and

$$\boldsymbol{\tau} \cdot \mathbf{D}[\mathbf{u}] \cdot \mathbf{n} = \overset{\circ}{\gamma}(r, z) \tag{6.6}$$

where the shear rate

$$\mathring{\gamma}(f(z), z) = \boldsymbol{\tau} \cdot \mathbf{D}[\boldsymbol{v}] \cdot \mathbf{n} \tag{6.7}$$

is evaluated on the solution $\boldsymbol{v}$ of equations (6.1)-(6.3). The constant β and function $\mathring{\gamma}(r, z)$ are prescribed.

After computing $\boldsymbol{v}$ and p_w from the problem (6.1) and $\mathbf{u}$ and p_0 from the problem (6.4)-(6.7), we may complete the perturbation cycle by forming the normal stress balance corresponding to (4.9). This balance replaces (6.2) with

$$\frac{\sigma}{f(1 - f'^2)^{\frac{1}{2}}} - \frac{\sigma f''}{(1 + f'^2)^{\frac{3}{2}}} - 2\mu_w \mathbf{D}[\mathbf{u}] \cdot \mathbf{n} = p_0 - p_w \tag{6.8}$$

Equation (6.8) cannot be satisfied for arbitrarily selected functions $r = f(z)$ and wave lengths L. These parameters are iterated at each perturbation cycle until (6.8) balances and holdupt ratio (6.3) is met, giving rise to converged values of $f(z)$ and L.

Preliminary calculations following the perturbation method just presented have been carried out using the methods of Bai et al [1996]. The additional terms in the normal stress balance (6.8) are small (see Figure 2) and the difference between the rigid approximation of Bai et al [1996] and the present calculation are also small (see Figure 3). The wavelength and pressure gradient versus Reynolds number is shown in Figure 4. The wavelength of perturbation is slightly larger than the wavelength of the rigid core, but the pressure gradients are the same.

A more complete study of the perturbation solution will be presented in a future calculation.

REFERENCES

BAI, R., KELKAR, K., & JOSEPH, D.D. 1996 Direct simulation of interfacial waves in a high viscosity ratio and axisymmetric core-annular flow. *J. Fluid Mech.* **240**, 97–142.
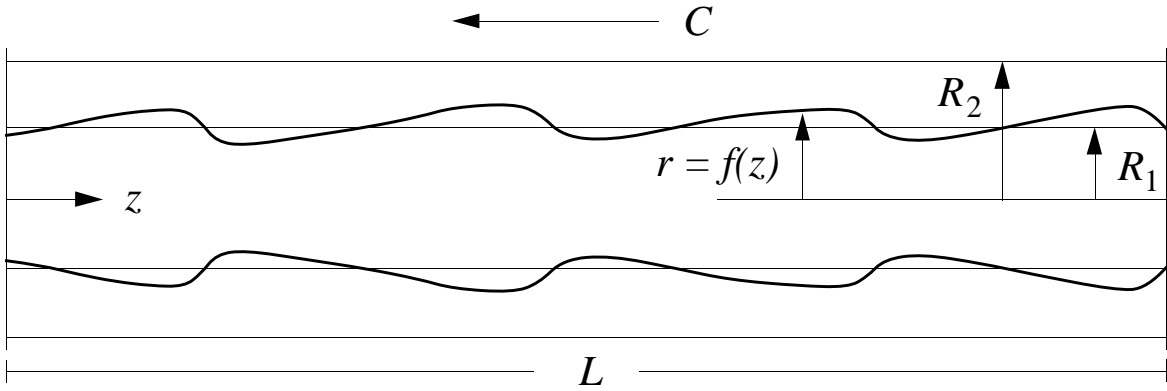
FIGURE 1.
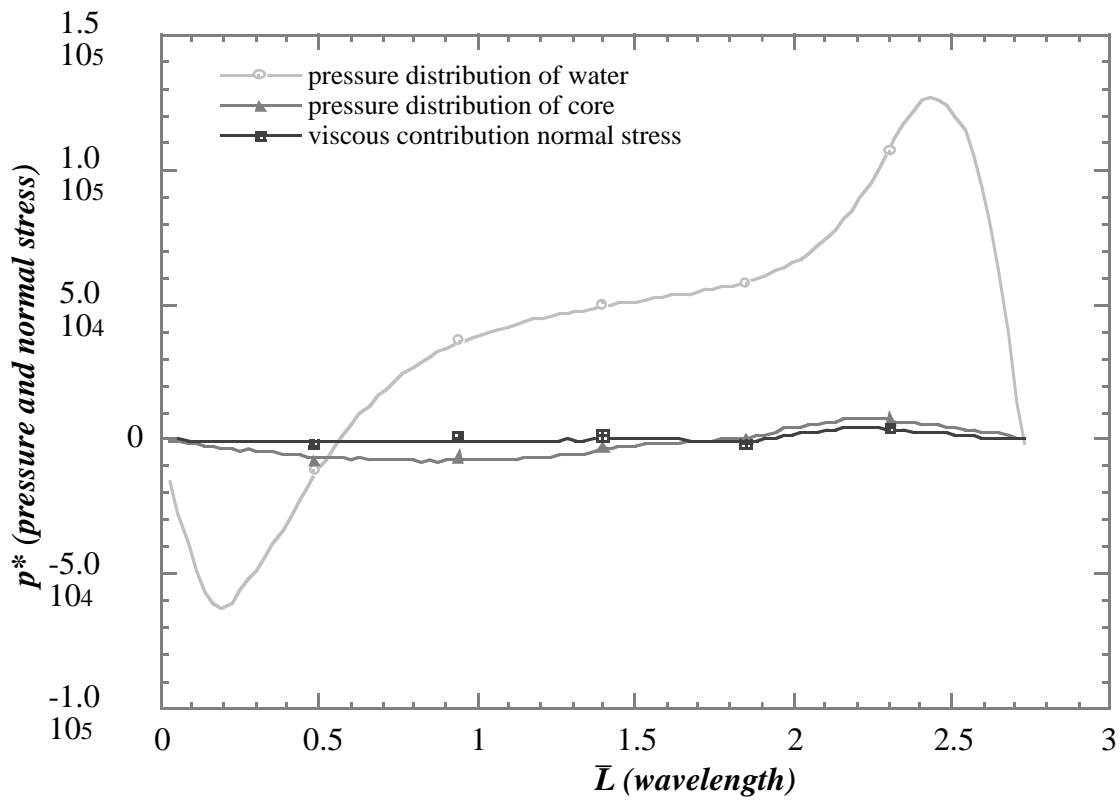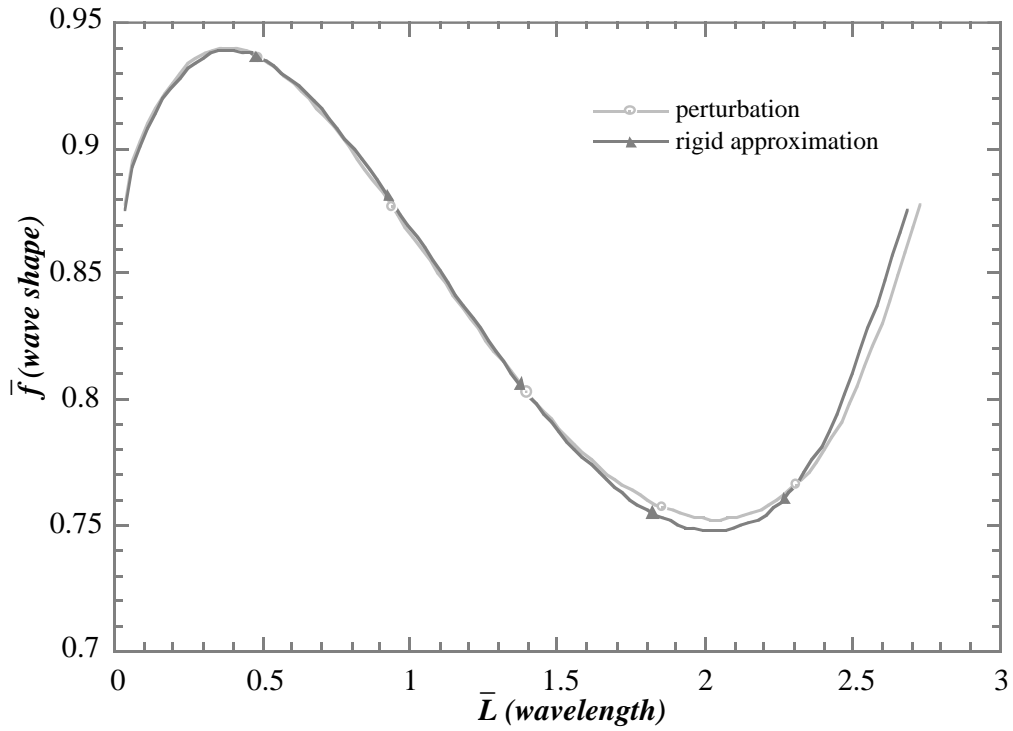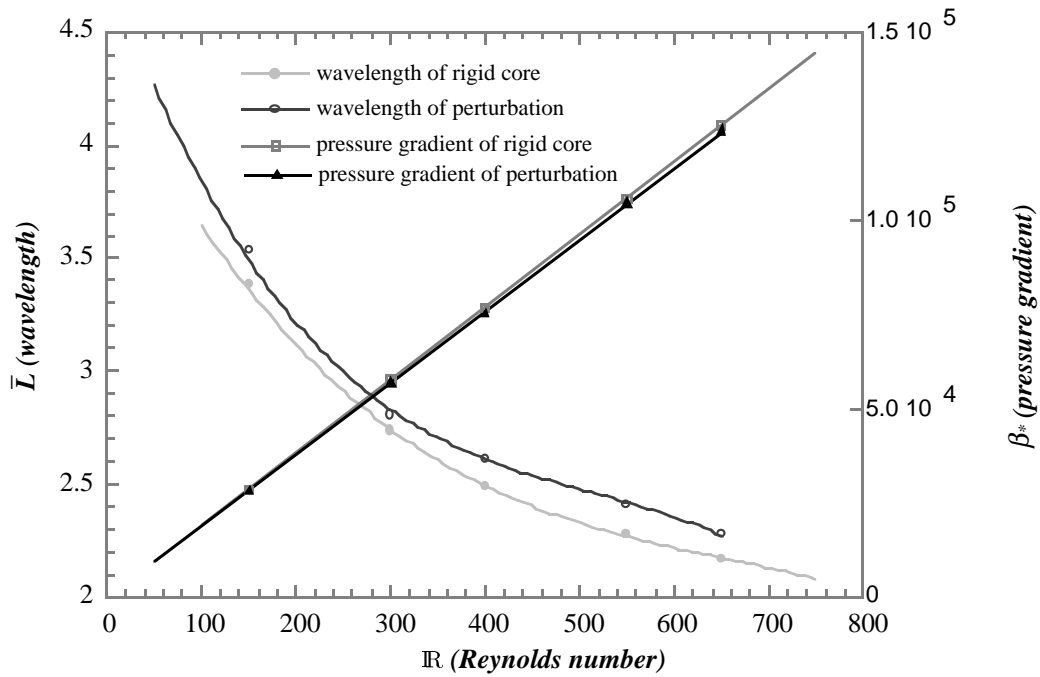

FIGURE 2.

FIGURE 3.

FIGURE 4.

Captions

Figure 1. Core annular flow. The flow is periodic with period L. The mean radius is R_1 where $R_1^2 = \frac{1}{L} \int_0^L f^2(z)$. The core moves forward with velocity c and the wall is stationary; here the core has been put to rest. Let Ω_w be the domain occupied by water $0 \leq z \leq L, f(z) \leq r \leq R_2$ and Ω_o is the domain occupied by oil $0 \leq z \leq L, 0 \leq r \leq f(z)$.

Figure 2. The pressure distributions and viscous contribution normal stress along the wave interface when $[\eta, h, \mathbb{R}, \mathrm{J}] = [0.8, 1.4, 600, 13 \times 10^4]$.

Figure 3. The comparison of the wave shapes approached from perturbation and rigid approximation when $[\eta, h, \mathbb{R}, \mathrm{J}] = [0.8, 1.4, 600, 13 \times 10^4]$.

Figure 4. The comparison of dimensionless wavelength and pressure gradient vs. Reynolds number $\mathbb{R}$ for $[\eta, h, \mathrm{J}] = [0.8, 1.4, 13 \times 10^4]$