Capillary collapse and rupture

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The breakup of a liquid capillary filament is analyzed as a viscous potential flow near a stagnation point on the centerline of the filament towards which the surface collapses under the action of surface tension forces. We find that the neck is of parabolic shape and its radius collapses to zero in a finite time; the curvature at the throat tends to zero much faster than the radius, leading ultimately to a microthread of nearly uniform radius. During the collapse the tensile stress due to viscosity increases in value until at a certain finite radius, which is about 1.5 microns for water in air, the stress in the throat passes into tension, presumably inducing cavitation there.

1. Introduction

The breakup of liquid jets is generally framed in terms of the capillary pressure $\sigma/R(z,t)$ due to surface tension σ acting at the neck of radius $R(z,t)$. The capillary pressure is greatest at the position z where R is smallest, an unstable situation in which liquid is squeezed out of the neck further reducing R and increasing the capillary pressure there. This picture leads to an inevitable collapse of the radius to zero. The conventional view is that the capillary instability just described leads to 'pinch-off' but the physics required to actually rupture the thread is not revealed. Here we are promoting the idea that the filament ruptures by cavitation due to tensile stresses induced by the motion out of the neck. The idea that liquids can cavitate by tensile stresses associated with motions, rather than by lowering the pressure was introduced by Joseph (1995). One of the interesting implications of this idea (Joseph 1998) is that cavitation in a pure shear flow may be induced by a tensile stress at 45° from the direction of shearing in a pure shear flow.

Capillary collapse is the final stage of dynamics which may be framed as starting from the capillary instability of a liquid cylinder. The initial instability of the liquid cylinder was studied by Funada & Joseph (2002) and Wang, Joseph & Funada (2005). Funada & Joseph (2002) assumed that the motion of the viscous cylinder is irrotational; the velocity is given by $\mathbf{u} = \boldsymbol{\nabla}\phi$, $\nabla^2\phi = 0$ and the viscous terms in the normal stress balances are evaluated from the potential. They derived a dispersion relation, σ vs. k, where σ is the growth rate and k the wave number. They compared growth rate curves for potential flows of inviscid and viscous fluids in which the conditions on the tangential components of the velocity and stress are neglected with the growth rates from the normal mode reduction of the Navier–Stokes equations (called exact) in which the vorticity and continuity of the tangential velocity and stress are not neglected. Many liquids with viscosities differing by many orders of magnitude were studied. In all cases there is a strict separation of the growth rate curves computed by three theories. The growth rates for inviscid potential flow are largest and those for the Navier–Stokes theory are smallest with viscous potential flow between. The curves are crowded when the viscosity is small and are widely separated when the viscosity is large. The potential flow solution for viscous fluids is in a modest agreement with the exact results, but the results for inviscid fluids are well off the mark.

Wang, Joseph & Funada (2005) implemented the method proposed by Joseph & Wang (2004) for computing a viscous correction of the irrotational pressure induced by the discrepancy between the non-zero irrotational shear stress and the zero shear stress boundary condition at a free surface. The theory with an additional viscous pressure correction added to the irrotational pressure is called the viscous correction of viscous potential flow (VCVPF). The corrected theory leads to a connection formula between the irrotational shear stress and the added viscous pressure which arises in a very thin boundary layer which is not analyzed and not needed. The linearized equations in this layer are used to show that the added pressure is harmonic and the additional contributions of the viscosity to the normal stress are small compared to the viscous irrotational contribution.

The analysis of capillary instability using the added pressure is in remarkable agreement with the results of exact analysis for all cases. The growth rate curves for VCVPF are nearly identical to those computed from the exact theory, uniformly in k. The two theories differ at most by a few percent whereas, for the case of highly viscous liquids, the analysis for inviscid liquids gives large unrealistic growth rates. The popular idea that viscous potential flow should be a small perturbation of inviscid potential flow is wrong.

The reader may think that the calculation of an added viscous pressure correction takes the theory away from purely irrotational flow, even though the velocities are obtained from the potential. However, exactly the same results that arise from VCVPF also arise from the dissipation method in which the pressure never enters; all the quantities needed are obtained from solutions of Laplace equation for the potential. The dissipation method used for the calculation of capillary instability is the strict analog of the dissipation method used by Lamb (1932) to determine the effects of viscosity on the decay of irrotational water and by Levich (1949) to determine the drag on a spherical gas bubble.

The problem of capillary collapse considered here is rather different than the problem of capillary instability of a liquid cylinder. One obvious difference is that the instability problem is linear but the collapse problem is very nonlinear. Less obvious is the role of the pressure correction which arises in the linear problem from the need to compensate the unbalanced irrotational shear stress. In the problem considered here the shear stress is continuous at the throat and the normal stresses are balanced there. A harmonic correction p_{v} of the irrotational pressure p_{i} is required to balance the normal stress away from the throat. This additional contribution to the pressure generates a vortical contribution to the velocity away from the throat.

2. Analysis

In this paper we study the collapse of a capillary filament under surface tension forces which squeeze liquid symmetrically from the throat at $z = 0$ in Figure 1. The analysis is local; terms of order z^4 are neglected but the local stagnation flow can be thought to be embedded in a global periodic structure of stagnation points with depletion at throats and accumulation at crests, as is shown in Figure 1.

We assume that the flow in the neighborhood of the throat is an axially symmetric straining flow, or stagnation point flow, with velocity components

$$u_z = a(t)\, z\,, \tag{2.1}$$

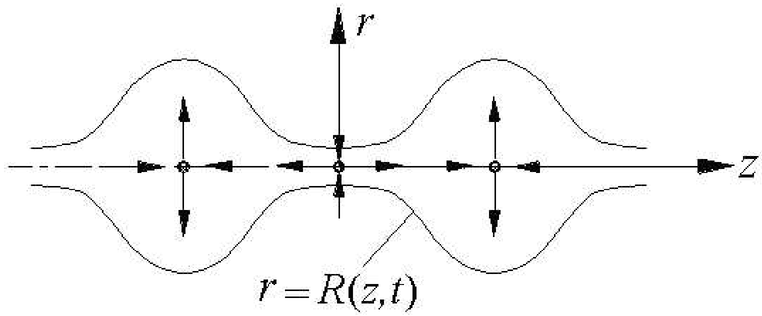

FIGURE 1. Periodic structure of stagnation points as a cartoon of the dynamics of capillary collapse. The collapse will give rise to a periodic string of liquid drops. The analysis here is local focusing on dynamics of collapse at $z = 0$.

$$u_r = -\frac{1}{2}a(t)\,r\,, \tag{2.2}$$

and determine the strain rate $a(t)$ and the capillary shape, $r = R(z,t)$, by satisfying the appropriate boundary conditions at the capillary surface. The velocity field described by equations (2.1) and (2.2) is incompressible and irrotational, therefore despite being a viscous flow, it may be described by a velocity potential ($u_z = \partial\phi/\partial z\,, u_r = \partial\phi/\partial r$) of form

$$\phi = \frac{1}{2}az^2 - \frac{1}{4}ar^2. \tag{2.3}$$

The pressure $p_{\rm i}$ in the flow is determined from the unsteady version of the Bernoulli equation

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\left({u_r}^2 + {u_z}^2\right) + \frac{p_{\rm i}}{\rho} = \frac{p_0}{\rho}\,, \tag{2.4}$$

in the form

$$\frac{p_{\rm i} - p_0}{\rho} = -\left(\frac{1}{2}\dot{a} + \frac{1}{2}a^2\right) + \left(\frac{1}{4}\dot{a} - \frac{1}{8}a^2\right)r^2. \tag{2.5}$$

The constant stagnation pressure p_0 may be related to a distant state of rest; p_0 is a global reference in an otherwise local solution. The overdot denotes a time derivative.

In this flow the state of stress is given by two principal stresses;

$$T_{zz} = -p_{\rm i} + 2\mu\frac{\partial u_z}{\partial z} = -p_{\rm i} + 2\mu a\,, \tag{2.6}$$

$$T_{rr} = -p_{\rm i} + 2\mu\frac{\partial u_r}{\partial r} = -p_{\rm i} - \mu a\,. \tag{2.7}$$

The normal traction at a point on the free surface, the force per unit area which the surface exerts on the fluid, is

$$T_{nn} = {n_r}^2 T_{rr} + {n_z}^2 T_{zz}\,, \tag{2.8}$$

where n_r and n_z are components of the unit outward normal. A force balance at the free surface gives the boundary condition

$$-T_{nn} - p_a = \sigma\kappa\,, \tag{2.9}$$

where p_a is atmospheric pressure, σ is the surface tension force per unit length and κ is the mean curvature, given by

$$\kappa = -\frac{\partial^2 R/\partial z^2}{\left(1+(\partial R/\partial z)^2\right)^{3/2}} + \frac{1}{R\left(1+(\partial R/\partial z)^2\right)^{1/2}}. \tag{2.10}$$

Equation (2.9) is the condition which drives the capillary collapse. It should be pointed out that the condition of zero shear stress at the boundary is satisfied exactly at the throat because u_z is independent of r.

Since the free surface must move with the fluid, we also have the kinematic condition

$$u_r = \frac{\partial R}{\partial t} + u_z \frac{\partial R}{\partial z} \tag{2.11}$$

at $r = R(z,t)$. This may be written

$$-\frac{1}{2} a R = \frac{\partial R}{\partial t} + az\frac{\partial R}{\partial z}. \tag{2.12}$$

The mathematical problem is to find a function $R(z,t)$ which satisfies the conditions expressed by equations (2.9) and (2.12). We will show that a function of form

$$R(z,t) = R_0(t) + R_2(t)\, z^2 + O(z^4) \tag{2.13}$$

is suitable and determine $R_0(t)$, $R_2(t)$ and the strain rate $a(t)$ by expanding these conditions for small z. To the lowest order in z^2

$$\begin{aligned} \frac{T_{nn}}{\rho} &= \frac{T_{rr}}{\rho} = -\frac{p_{\mathrm{i}}}{\rho} - \nu a \\ &= -\frac{p_0}{\rho} - \nu a + \frac{1}{2}\left(\dot a + a^2\right) z^2 - \left(\frac{1}{4}\dot a - \frac{1}{8}a^2\right) R^2 \\ &= -\frac{p_0}{\rho} - \nu a + \frac{1}{2}\left(\dot a + a^2\right) z^2 - \frac{1}{4}\left(\dot a - \frac{a^2}{2}\right)\left({R_0}^2 + 2R_2 z^2\right) \end{aligned} \tag{2.14}$$

and to the same order

$$\kappa = \frac{1}{R_0} - 2R_2 . \tag{2.15}$$

Equation (2.12) gives the two equations

$$-\frac{1}{2} a R_0 = \dot R_0 , \tag{2.16}$$

$$-\frac{5}{2} a R_2 = \dot R_2 . \tag{2.17}$$

From these, we see that $R_2 = C{R_0}^5$, where C is a constant depending on starting conditions. This result implies that R_2 tends to zero faster than R_0 which means that the parabola flatters out during collapse. It follows then that the R_2 term in (2.14) is of lower order and the term proportional to z^2 can not be balanced.

To balance the terms proportional to z^2 in (2.14), we introduce a pressure correction p_{v} where $p = p_{\mathrm{i}} + p_{\mathrm{v}}$. We find this correction among harmonic functions $\nabla^2 p_{\mathrm{v}} = 0$ so that

$$\nabla^2 p = \nabla^2 p_{\mathrm{i}} = -\rho \operatorname{div}\left(\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}\right),$$

where $\mathbf{u} = \boldsymbol{\nabla}\phi$ and p_{i} is given by (2.5). The required harmonic function is found in the form

$$p_{\mathrm{v}} = C\left(-\frac{r^2}{2} + z^2\right), \tag{2.18}$$

and, after adding p_{v} to (2.5), we get

$$\frac{p-p_0}{\rho} = -\left(\frac{1}{2}\dot{a} + \frac{1}{2}a^2 - \frac{C}{\rho}\right)z^2 + \left(\frac{1}{4}\dot{a} - \frac{1}{8}a^2 - \frac{C}{2\rho}\right)r^2.$$

To balance the terms proportional to z^2 in (2.14), we choose

$$C = \frac{\rho}{2}\left(\dot{a} + a^2\right).$$

Then

$$\frac{p-p_0}{\rho} = -\frac{3}{8}a^2 r^2. \tag{2.19}$$

This pressure difference is negative and is most negative at the boundary $r = R$.

The pressure correction induces a vortical velocity $\mathbf{v}$ which vanishes at the throat. The velocity $\mathbf{u} = \mathbf{u}_{\mathrm{i}} + \mathbf{v}$, where the components of $\mathbf{u}_{\mathrm{i}} = \boldsymbol{\nabla}\phi$ are given by (2.1) and (2.2), and

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v} + \mathbf{u}_{\mathrm{i}}\cdot\boldsymbol{\nabla}\mathbf{v} + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{u}_{\mathrm{i}} = -\boldsymbol{\nabla}\frac{p_{\mathrm{v}}}{\rho} + \nu\nabla^2\mathbf{v}. \tag{2.20}$$

We shall show that the left side of (2.20) is of lower order and may be neglected near the stagnation point. Writing $(v_r, v_z) = (u, w)$, and using (2.18), we have

$$\begin{aligned} 2Cz &= \mu\left[\frac{\partial^2 w}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\right], \\ -Cr &= \mu\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{u}{r^2}\right], \\ \frac{\partial w}{\partial z} &+ \frac{1}{r}\frac{\partial}{\partial r}(ru) = 0. \end{aligned} \tag{2.21}$$

The solution of (2.21) which vanishes at the origin is

$$w = \frac{C}{3\mu}z^3, \qquad u = -\frac{C}{2\mu}rz^2. \tag{2.22}$$

The vorticity for this axisymmetric solution is given by

$$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} = -\frac{C}{\mu}rz. \tag{2.23}$$

The largest terms on the right side of (2.20) for values of r and z are $O(z^3)$ and $O(rz^2)$. The vortical velocity $\mathbf{v}$ does not enter into any leading balance discussed below.

To leading order, $T_{nn} = T_{rr} = -p + \mu a$ and $-T_{rr} - p_a = \sigma/R_0$, which is in the form

$$-\frac{3\rho}{8}\dot{R_0}^2 - \mu a + p_0 - p_a = \frac{\sigma}{R_0}. \tag{2.24}$$

Since we are most interested in small R_0, we can pick out the dominant terms in (2.24) as R_0 tends to zero. These are

$$-2\mu\frac{\dot{R}_0}{R_0} = \frac{\sigma}{R_0},$$

which is a balance between the viscous part of the normal force (which resists the collapse) and the surface tension force (which drives it). The large ${R_0}^{-1}$ term cancels from each side giving

$$\dot{R}_0 = -\frac{1}{2}\frac{\sigma}{\mu}, \tag{2.25}$$

with solution

$$R_0 = \frac{\sigma}{2\mu}(t_* - t) , \tag{2.26}$$

where t_* is a constant of integration. Therefore we have a solution in which R_0 tends to zero in a finite time. By expanding R_0 in powers of $(t_* - t)$ it is easy to see that the neglected terms in equation (2.24) give a correction to R_0 of order $(t_* - t)^2$. With the additional term the solution becomes

$$R_0 = \frac{\sigma}{2\mu}(t_* - t) - \frac{\sigma}{2\mu}\frac{p_0 - p_a}{4\mu}(t_* - t)^2 + \dots \quad . \tag{2.27}$$

The strain rate is,

$$a = -2\frac{\dot{R}_0}{R_0} = \frac{2}{t_* - t} - \frac{p_0 - p_a}{2\mu} + \dots , \tag{2.28}$$

The axial stress at leading order is given by

$$\begin{aligned} T_{zz} &= -p + 2\mu a = -p_0 + \frac{3}{8}\rho a^2 r^2 + 2\mu a \\ &= -p_0 + \frac{3}{2}\rho \dot{R}_0^2 \left(\frac{r}{R_0}\right)^2 + 2\mu a \\ &= \frac{3}{8}\frac{\rho\sigma^2}{\mu^2}\left(\frac{r}{R_0}\right)^2 + \frac{4\mu}{t_* - t} - (2p_0 - p_a) , \end{aligned} \tag{2.29}$$

where $\dot{R}_0^2 = \sigma^2/4\mu^2$, from (2.25). The stress induced cavitation will occur when and where the axial stress passes into tension. This will always occur first at the boundary of the capillary where $r/R_0 = 1$.

Consider next the axial stress at the stagnation point $(r = 0, z = 0)$

$$T_{zz} = -(2p_0 - p_a) + \frac{2\sigma}{R_0(t)} . \tag{2.30}$$

The thread will pass into tension over the whole cross section at $z = 0$ when T_{zz} given by (2.30) becomes positive and passes into tension.

We see that when R_0 is sufficiently small T_{zz} can become positive. This means that the axial stress becomes tension instead of compression. Liquids can not support much tension without rupturing. This would occur here when R_0 is somewhat less than the critical value

$$R_{0cr} = \frac{2\sigma}{(2p_0 - p_a)} , \tag{2.31}$$

which, it should be noted, is independent of the viscosity. This value is fairly large; for water with $\sigma = 75$ dynes/cm and estimating p_0 to be $p_0 \approx p_a = 10^6$ dynes/cm^2, we get $R_{0cr} = 1.5$ microns.

A Reynolds number for the collapsing capillary may be defined by

$$Re = \frac{R_0\dot{R}_0}{\nu} , \tag{2.32}$$

based on the throat radius and the velocity of collapse, using equation (2.25) for the latter quantity gives

$$Re = \frac{R_0 \sigma}{2\rho\nu^2}\,, \tag{2.33}$$

which is the ratio of R_0 to a viscous length (Peregrine, Shoker & Symon 1990) $2\rho\nu^2/\sigma$ which is very small for water, about 0.027 microns. Therefore using R_{0cr} (= 1.5 microns) for R_0, the Reynolds number at collapse is about 55 for water (the collapse velocity is about 37 m/s). For more viscous liquids the Reynolds number at collapse could be very small. For the solution presented here there is no restriction on the magnitude of the Reynolds number.

The symmetric local solution derived here may not be stable; photographs of breaking liquid bridges (Peregrine *et al.* 1990) are globally asymmetric. A strongly collapsing capillary could be expected to amplify asymmetries, as is known to happen in a collapsing bubble. Nevertheless, the local symmetric solution presented here is of interest (as is spherical bubble collapse) and the fracture at a finite value of the radius due to viscous stresses is perhaps independent of the global properties of the solution.

3. Conclusions and discussion

Neckdown of a liquid capillary thread was studied in a local analysis based on viscous potential flow. One objective of this study was to show that during collapse the thread will enter into tension due to viscosity and can be expected to fracture, or cavitate, at a finite radius.

The flow in the throat of the collapsing capillary is locally a uniaxial extensional flow, linear in z and r , with a time dependent strain rate $a(t)$. This viscous potential flow satisfies the Navier–Stokes equation and all the relevant interfacial conditions, including continuity of the shear stress. The principal dynamic balance is between the surface tension forces, which are trying to collapse the capillary, and the radial viscous stress which is resisting the collapse. Since mass must be conserved a large axial flow results from squeezing liquid out of the neck and this results in a large viscous extensional stress. The extensional stress passes into tension at $R_0 = 1.5$ micron (for water and air) long before R_0 actually collapses to zero. The solution is symmetric about $z = 0$, the position of the smallest radius; the axial velocity is odd and the radial velocity, pressure and interface shape

$$R(z,t) = R_0(t) + R_2(t)\, z^2 + O(z^4)\,,$$

are even in z. At lowest order the interface is a parabola in which $R_2(t)$ is proportional to ${R_0}^5$, hence in the limit of collapsing radius $R_2 \to 0$ much more rapidly than R_0 and the shape approaches that of a straight cylinder. The radius tends to zero linearly, like $(t_* - t)$, collapsing to zero in a finite time. At the same time the strain rate $a(t)$ tends to infinity like $(t_* - t)^{-1}$.

In the literature on capillary collapse and rupture the focus is on collapse which is universally framed as a 'pinch-off' and the fundamental physics governing the rupture of the thread is not considered. A 'pinch-off' is a squeezing flow; the radius of the jet at the pinch point collapses, squeezing fluid out as the filament collapses. Here one finds a stagnation point; stagnation point flow is a potential flow and the effects of viscosity in such a flow may be huge. Certainly, potential flow of an inviscid fluid is not the right tool here. We can get results in which viscosity acts strongly using viscous potential flow. The question is not whether viscosity is important, which it is, but whether vorticity is important.

Chen, Notz & Basaran (2002) have studied 'pinch-off' and scaling during drop formation using high-accuracy computation and ultra-fast high-resolution imaging. They discuss dynamic transition from potential flow with a 2/3 scaling due to Keller & Miksis (1983) to an inertial-viscous regime described by Eggers (1993) universal solution. They find overturn before breakup in experiments in water (1 cp) well before the dynamic transition from the potential flow to the inertial-viscous regime. On the other hand, an 85 cp glycerol-water solution is said to exhibit this transition. The potential flow solutions discussed by Chen *et al.* 2002 are for inviscid solutions. Of course, water and glycerol are not inviscid. The scaling of Keller & Miksis (1983) which gives rise to the 2/3 power collapse law does not work for viscous potential flow. The spoiler is their equation (3.3) expressing the normal stress balance. To this equation we must add the viscous component $2\mu\partial u_n/\partial n$. The term $|\nabla\phi|^2$ in (3.3) scales like ϕ^2/L^2 whereas the viscous component scales like ϕ/L^2, so that the similarity transformation does not factor through. Analogies have been put forward between capillary 'pinch-off' of a viscous fluid thread and van der Waals driven ruptures of a free thin viscous sheet by Vaynblat, Lister & Witelski (2001). The observation that a filament under capillary collapse ruptures in a 'pinch-off' does not come to grips with the physics which leads to a loss of the continuum. One idea is that thread breaks under the action of disjoining pressures. Unfortunately, a mathematical theory for disjoining pressures for thin threads is not available.

The recent literature on capillary collapse is presently dominated by the discovery of self-similar, finite time singularity formation. These solutions are discussed in the recent papers Chang, Demekhin & Kalaidin (1999), McKinley & Tripathi (2000), and in the paper of Chen *et al.* (2002). This literature does not treat the physics of rupture or breakup by cavitation and does not compute stresses. All of the above mentioned authors find that capillary radius decreases to zero linearly in time, but the rate of collapse differs from author to author. McKinley & Tripathi (2000) write the formula

$$R_{\mathrm{mid}}(t) = R_1 - \frac{2X-1}{6}\frac{\gamma}{\mu}t\,,$$

for the neck radius of the collapsing capillary in the stage of final decay as t increases to t_* when $R_{\mathrm{mid}}(t_*) = 0$. They give the X obtained by different authors in their Table 1, but without the value $X = 2$ obtained here for viscous potential flow, giving the fastest decay. Eggers (1997, 1993) obtained $X = 0.5912$ and Papageriou (1995) obtained $X = 0.7127$. The solutions of the two authors last named have vorticity; Papagerious solution has no inertia. McKinley & Tripathi (2000) note that very close to breakup the solution of Papageriou crosses over to Eggers similarity solution. The transitions between different similarity solutions are less well understood than the similarity solutions themselves. These transitions can be regarded as a form of instability. It is possible that the solution in this paper can be described as a transition to rupture.

The solution of Eggers (1993) gives rise to a universal scaling law which has been observed for viscous liquids but not in water (see Chen *et al.* 2002). The long wave approximation used to derive universal scalings may prevent it from resolving the dynamics of rupture. The similarity solution of Eggers does not lead to a cavitation threshold; in his solution the tension due to extension increases but not fast enough to overcome the compression due to the capillary pressure of the thinning filament.

The criterion for the termination of the continuum is probably not a finite time singularity; the thread radius does not go to zero. It comes apart before then.

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