J. Fluid Mech. (2009), *vol.* 633, *pp.* 475–483.
doi:10.1017/S0022112009990632 Printed in the United Kingdom

Particle-wall collision in a viscoelastic fluid

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(Received 8 September 2008 and in revised form 28 May 2009)

In this study, we present experimental results on particle-wall collision in viscoelastic fluids. A sphere is released in a tank filled with poly(ethylene-oxide) (PEO) mixed with water with varying concentrations up to 1.5 %. The effect of Stokes and Deborah numbers on the rebound velocity of a spherical particle colliding onto a wall is considered. It has been observed that the slope at which the coefficient of restitution increases with Stokes number is smaller for higher Deborah numbers. Higher rebound occurs for higher PEO concentration at the same stokes number. However, the results for the coefficient of restitution in polymeric liquids can be collapsed together with the Newtonian fluid behaviour if one defines the Stokes number based on the local strain rate.

1. Introduction

Particle–particle and particle-wall collisions occur in many natural and industrial applications such as sedimentation, agglomeration, granular flows and, in general, any multi-phase flow application involving particles. To accurately predict the behaviour of particulate flows, fundamental knowledge of the mechanisms of single-particle collision is required. More specifically, the study of particle-wall collisions provides deeper insight into modelling particle-laden flow when particle interaction is important. However, almost all of the existing studies on particle collision focus on Newtonian fluids. In this paper, we experimentally study particle-wall collision in viscoelastic liquids.

Several studies on the rebound of colliding particles in Newtonian fluids have been conducted during the last three decades. Davis, Serayssol & Hinch (1986) employed an elastohydrodynamic model for collision between particles suspended in a liquid. Throughout these collisions, the particles are separated by a viscous liquid film. The pressure force in this lubrication film is sufficiently large to cause the particles to deform and rebound without making solid-body contact. It was shown that the pertinent parameter for collision in the fluid is not the Reynolds number Re but the Stokes number $St = (1/9)(\rho_p/\rho_f)Re$ where ρ_p and ρ_f are the particle and fluid densities, respectively. No rebound occurs for St lower than a critical value due to the fact that the elastic energy stored by the particle deformation is dissipated in the fluid. It has been shown that the coefficient of restitution normalized by that for a dry

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collision depends strongly on the impact Stokes number and weakly on the elastic properties of the particle, where the Stokes number is defined using the approaching velocity of the particle. It has been shown that below a St of 10, no rebound occurs (e.g. Gondret *et al.* 1999; Joseph *et al.* 2001; Gondret, Lance & Petit 2002). For impact Stokes numbers larger than 500 the coefficient of restitution asymptotes to that for a dry collision. Davis *et al.* (1986) concluded that the Stokes number dependence is due to the drainage of the liquid film formed between the particle and the wall as noted by Legendre *et al.* (2006). Barnocky & Davis (1989) considered the variation of the density and viscosity with pressure. The increase in viscosity and density with pressure leads to solidification of the fluid in the contact region which affects the rebound. However, they concluded that the effect of the increase in viscosity on the normal collision behaviour is small (noted by Joseph & Hunt 2004). Davis *et al.* (1986) showed that the elasticity parameter $\varepsilon = (4\theta\mu V_i a^{3/2})/h^{5/2}$ affects the rebound velocity of particles, where h is the gap between the two approaching surfaces and the relative velocity is V_i; $\theta = (1-\nu_1^2)/\pi E_1 + (1-\nu_2^2)/\pi E_2$ and ν_1, ν_2 and E_1, E_2 are the Poisson's ratios and Young's moduli of elasticity for the particle and the wall respectively; $a = d/2$ is the particle radius.

Surface roughness has a significant effect on rebound velocity because the lubrication layer between two colliding objects is very small and may be of the same order as the surface roughness. Consequently, contact may occur through microscopic surface imperfections, as noted by Smart & Leighton (1989). Davis (1987) developed a theory for collision of rough surfaces with small bumps with dilute surface coverage. He showed that the surface roughness has negligible effect on the viscous force until the gap between the smooth surfaces becomes equal to the size of largest roughness element. At this time, the bumps make physical contact due to the discrete molecular nature of the fluid and/or attractive London–van der Waals forces. Further approach is thereby prevented and solid-solid contact occurs. Ardekani & Rangel (2008) recently studied the effects of surface roughness and Stokes number on the rebound velocity of a bouncing particle on a wall in a viscous fluid. Their numerical results agree with the experimental results by Gondret, Lance & Petit (2002). Their method has been extended to include multi-particle collision and the collision of general shape objects (Ardekani, Dabiri & Rangel 2008*a*).

Whereas several experimental studies have been conducted of the influence of the Newtonian fluid properties on collision processes, only a few studies address particle collision in viscoelastic fluids. Stocchino & Guala (2005) studied particle-wall collision in a shear-thinning fluid and observed that the coefficient of restitution in the case of non-Newtonian fluids (aqueous solution of carboxymethyl cellulose) is higher compared to the Newtonian case for the same Stokes number. Guala & Stocchino (2007) provided particle image velocimetry measurements of the velocity field during rebound of steel particles in the same liquid and concluded that at low Deborah numbers, which represents the ratio between a characteristic relaxation time and the characteristic time scale of the experiment, the shear-thinning character of the non-Newtonian fluid is dominant with respect to its viscoelasticity. Ardekani, Rangel & Joseph (2007) studied the normal motion of a spherical particle towards a wall in a second-order fluid and observed that the contribution of the second-order fluid to the overall force applied to the particle is an attractive force towards the wall. For a particle moving towards a wall in a second-order fluid of both Stokes and slow potential flows, a smaller drag force is experienced by the particle as compared to the Newtonian case. Ardekani, Rangel & Joseph (2008*b*)'s interpretations of the aggregation of particles in viscoelastic fluids rest on three pillars. The first is the

No.	Material	d (mm)	ρ_p (g cm^{-3})	m (gr)	E (Gpa)	ν
1	Steel	2.381	7.67	0.054	203	0.29
2	Steel	3.175	7.67	0.128	203	0.29
3	Steel	4.762	7.67	0.434	203	0.29
4	Steel	6.350	7.67	1.03	203	0.29
5	Steel	7.936	7.67	2.01	203	0.29
6	Steel	9.525	7.67	3.47	203	0.29
7	Steel	12.700	7.67	8.22	203	0.29
8	Steel	19.050	7.67	27.753	203	0.29
9	Lucite (bottom wall)	50.8 (thickness)	1.18	–	2.93	0.35–0.4

TABLE 1. Particle and the wall properties.

existence of a viscoelastic 'pressure' generated by normal stresses due to shear. Second, the total time derivative of pressure is an important factor in the force applied to a moving particle. The third is associated with a change in the sign of the normal stress at points of stagnation. This is a purely extensional effect unrelated to shearing. These interpretations are suggested by analysis of a second-order fluid which arises asymptotically for motions which are slow and slowly varying.

In this study, the particle-wall interaction in viscoelastic fluids is experimentally studied. The motion of particles in a viscoelastic liquid can be expressed in terms of the Reynolds Re and Deborah De numbers. The Reynolds number is given by

$$Re = \frac{\rho_f V d}{\mu}, \tag{1.1}$$

where V is the terminal velocity of the particle and d is the particle diameter. The Deborah number is defined as

$$De = \frac{\lambda_0 V}{d}, \tag{1.2}$$

where λ_0 is the relaxation time. The relaxation time $\lambda_0 = \mu_0/\rho_f c^2$ is taken from a wave-speed measurement where c is the shear wave speed (Joseph, Riccius & Arney 1986; Riccius, Joseph & Arney 1987). For particle-wall collision in a viscoelastic fluid, the important parameters are the Stokes and Deborah numbers based on impact velocity rather than particle terminal velocity. The main purpose of the present work is to study whether the viscoelastic properties of the liquid may influence the measured coefficient of restitution. The experimental set-up is described in § 2 and the experimental results are discussed in § 3.

2. Experimental set-up

In this experiment, spherical particles are dropped in a liquid-filled tank. The spheres are released by means of a magnetic device located in the fluid approximately 15 cm above the bottom of the tank similarly to the experiment by Stocchino & Guala (2005) (their figure 1). The particle parameters are shown in table 1. The container is a hexagonal perspex tank with an edge length of 11 cm. The dimensions of the tank are such to avoid any influence of the sidewalls. A block of lucite with 50.8 mm thickness is used as the bottom wall of the tank. The temperature of the room is 24 ± 1 °C.

No.	ϕ (%)	T (°C)	n_1 (g cm^{-1})	c (cm s^{-1})	μ_0 (poise)	k	n	λ_0 (s)
1	0.5	24	124	9.05	5.2	5.17	0.537	0.06
2	0.6	24	162	10.0	7.43	6.76	0.503	0.07
3	0.75	24	208	11.9	25.2	20.3	0.463	0.18
4	1	24	360	14.0	76.5	39.7	0.419	0.39
	1	35	360	–	–	–	–	–
	1	45	321	–	–	–	–	–
	1	54	301	–	–	–	–	–
5	1.5	24	440	20.3	204	101	0.378	0.5

TABLE 2. Physical properties of aqueous Polyox (WSR-301, molecular weight 4×10^6) by Liu & Joseph (1993).

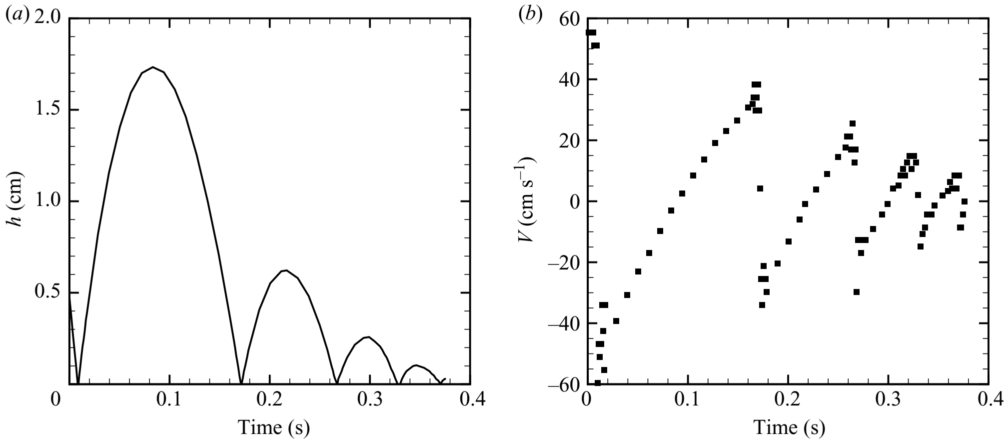

FIGURE 1. Collision of a 12.7 mm particle onto a wall in 1 % poly(ethylene-oxide) (PEO).

A mixture of up to 80 % glycerol in water and up to 1.5 % aqueous solution of poly(ethylene-oxide) (PEO) Polyox WSR-301 is used as the fluid for the experiments. The concentration (ϕ) by weight of the polymer was varied from 0.5 %–1.5 %. The liquid parameters, such as the zero shear value of the first normal stress coefficient n_1, the shear wave speed c from which relaxation time can be calculated, the zero shear rate viscosity μ_0 and the power law constant k and n are given in table 2. From the power law equation we have $\mu = k\dot{\gamma}^{n-1}$ where $\dot{\gamma}$ is the shear rate.

The motion of the sphere is captured using a high-speed digital camera (HCC-1000 512MB) with framing rates up to 2000 frames per second (f.p.s.). Most of the experiments carried out in this paper are captured at 912 f.p.s.. The digital images are processed to determine the position of the centroid of the sphere in each frame. The overall image size is 1024×256 pixels. The precision of the position can be determined within 0.7 % of particle diameter, corresponding to a resolution of one pixel. Figure 1 shows the trajectory and velocity of a sphere sedimenting in a 1 % Polyox solution during successive rebounds. Here, h is the separation distance between the surface of the particle and the wall. Two lines are drawn through the five data points before and after collision as done by Joseph *et al.* (2001). The slope of the fitted lines correspond to V_i and V_r, the impact and rebound velocity, respectively. The coefficient of restitution is defined as $e = -V_r/V_i$.

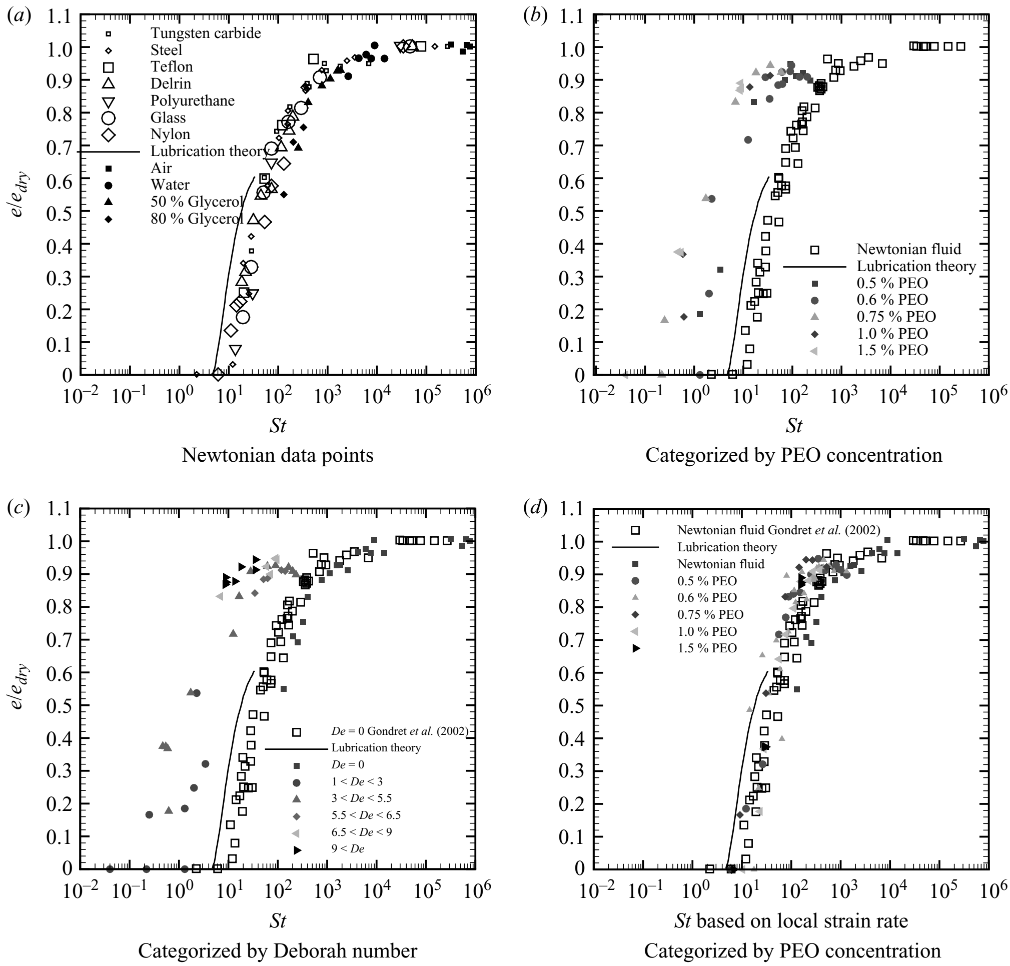

FIGURE 2. Coefficient of restitution normalized by that for dry collision as a function of St. Present results are shown by solid symbols. Experimental measurements for different materials by Gondret *et al.* (2002) are shown by open symbols. Lubrication theory of Davis *et al.* (1986) (—).

The coefficient of restitution corresponding to the collision of steel spheres onto a lucite wall in air e_{dry} is 0.93 ± 0.02. Surface roughness also has an important effect on rebound velocity as studied earlier by Joseph *et al.* (2001); Ardekani & Rangel (2008). Experimental results by Joseph *et al.* (2001) indicated that the characteristic variance observed in measurement of the coefficient of restitution is of the order of the experimental uncertainty for smooth particles and considerably larger for the rough particles. In this experiment, the particle roughness height is small as determined by scanning electron micrographs of the spheres at a few random spots showing that the particle roughness is smaller than 0.5 μm.

3. Experimental results and discussion

Figure 2(*a*) shows the coefficient of restitution for rebounding spheres in the Newtonian fluids listed in table 3. The results are compared with those by Gondret

Fluid	ρ_f (g cm^{-3})	μ (g cm^{-1}s^{-1})
Air	1.2×10^{-3}	1.85×10^{-4}
Water	1.0	0.00897
50 wt % glycerol	1.1	0.0432
80 wt % glycerol	1.18	0.369
Glycerol	1.26	7.140

TABLE 3. Physical properties of Newtonian liquids at 25°C by Shankar & Kumar (1994).

Fluid	d (mm)	St	De	e/e_{dry}	Fluid	d (mm)	St	De	e/e_{dry}
1.5 % PEO	19.05	8.62	16	0.87	0.6 % PEO	9.52	95	6.7	0.94
1.5 % PEO	19.05	8.84	16	0.89	0.6 % PEO	7.94	62	6.4	0.89
1.5 % PEO	19.05	9.09	17	0.87	0.6 % PEO	6.35	28.2	5.1	0.91
1.5 % PEO	12.70	0.47	4.4	0.37	0.6 % PEO	4.76	2.28	1.4	0.54
1.5 % PEO	12.70	0.54	4.8	0.37	0.6 % PEO	3.17	1.31	1.6	0.00
1.5 % PEO	9.52	0.04	1.3	0.00	0.6 % PEO	12.7	90	4.4	0.93
1 % PEO	19.05	35.3	19	0.91	0.6 % PEO	9.52	61.6	5.0	0.92
1 % PEO	12.70	13.6	17	0.88	0.6 % PEO	7.94	51.46	5.6	0.88
1 % PEO	9.52	0.59	3.4	0.37	0.6 % PEO	6.35	34.4	5.8	0.84
1 % PEO	7.94	0.62	4.4	0.18	0.6 % PEO	4.76	12.5	4.3	0.72
0.75 % PEO	19.05	59.1	8.6	0.92	0.6 % PEO	3.17	2.01	2.2	0.25
0.75 % PEO	12.70	35.5	10	0.94	0.5 % PEO	19.05	234	3.8	0.90
0.75 % PEO	9.52	18.1	9.8	0.92	0.5 % PEO	12.70	165	5.2	0.92
0.75 % PEO	7.94	6.92	6.7	0.83	0.5 % PEO	9.52	118	6.1	0.91
0.75 % PEO	6.35	1.73	3.6	0.54	0.5 % PEO	7.94	90	6.5	0.95
0.75 % PEO	4.76	0.25	1.5	0.17	0.5 % PEO	6.35	71	7.5	0.90
0.75 % PEO	3.17	0.22	2.3	0.00	0.5 % PEO	4.76	16.5	4.1	0.83
0.6 % PEO	19.05	200	4.3	0.91	0.5 % PEO	3.17	3.43	2.4	0.32
0.6 % PEO	12.70	140	5.9	0.91	0.5 % PEO	2.38	1.31	1.9	0.19

TABLE 4. Coefficient of restitution, Stokes number and Deborah number for different measurements.

et al. (2002). Based on the accuracy of the results, we proceed to calculate the coefficient of restitution for rebounding spheres in a PEO aqueous solution.

Table 4 shows the coefficient of restitution for the first rebound, Stokes number, and Deborah number for different measurements. The coefficient of restitution as a function of Stokes number for different poly(ethylene-oxide) (PEO) concentrations is plotted in figure 2(*b*). It should be noted that figures 2(*a*)–2(*c*) include only the data for the first rebound. Whereas, figure 2(*d*) includes the data for the first rebound as well as the successive ones. For the aqueous solution of PEO, μ varies with shear rate $\mu = \mu(\dot{\gamma})$ as indicated earlier. As explained by Guala & Stocchino (2007), it is necessary to assign a specific value to the rate of strain $\dot{\gamma}$ in order to evaluate the viscosity from the constitutive law. Mena, Manero & Leal (1987) studied the effect of the rheological properties on the drag force experienced by a settling particle. They suggested that for low *De* numbers, the shear-thinning behaviour can be represented with a Newtonian fluid model using the non-Newtonian viscosity corresponding to a strain rate $\dot{\gamma} = V_t/d$, where V_t is the particle terminal velocity. However, for intermediate *De* numbers, elastic effects are primarily important. For the impact problem, the impact velocity V_i should be used instead of the terminal velocity. Stocchino & Guala (2005) showed that for shear-thinning liquids, the use of V_i/d for the effective strain rate results in a higher coefficient of restitution as compared to the Newtonian case. In table 4 and figure 2(*b*), $\dot{\gamma} = V_i/d$ is used as the effective strain rate. As shown in this figure, a higher coefficient of restitution occurs in the PEO solution when compared to Newtonian fluids, and the coefficient is higher for higher PEO concentration. Both the shear-thinning and viscoelasticity affect the coefficient of restitution and it is difficult to discern the effect of each property. In addition, as seen in figure 2(*b*), the critical Stokes number below which no rebound occurs, decreases as the concentration of PEO increases. Figure 2(*c*) shows the data provided in table 4 but this time categorized with respect to the Deborah number. As it can be seen, higher rebound occurs for higher *De*. Higher Deborah numbers correspond to higher polymer concentration, higher viscoelasticity and higher shear-thinning effects. In figure 2(*c*), we observe that the slope at which the coefficient of restitution increases with *St* is smaller for larger Deborah number. Furthermore, smaller critical Stokes numbers are observed for larger *De*.

The force experienced by a particle moving towards a wall in a power-law fluid under the lubrication assumption can be written as $Fa/mV_i^2 = (a/h)(1/St)f(n)$ where m is the mass of the particle and $f(n)$ is a function of n defined as

$$\frac{2^{(3+n)/2}}{6}\left(\frac{2n+1}{n}\right)^n \beta\left(\frac{n+3}{2}, \frac{3n-1}{2}\right)$$

(Rodin 1996); n is the exponent that relates the shear stress and the corresponding strain rate ($\tau \propto \dot{\gamma}^n$); β is the beta function and $St = 2\rho_p V_i a/9k(\frac{V_i\sqrt{a}}{h^{3/2}})^{n-1}$. This expression shows that for a purely shear-thinning liquid, defining the effective shear rate as $V_i\sqrt{a}/h^{3/2}$ results in the collapse of the data on the Newtonian behaviour ($Fa/mV_i^2 = (a/h)(1/St)$) if $f(n)$ changes weakly with n. In this experiment $f(n)$ changes between 1 for Newtonian fluid and 13.6 for 1.5 % PEO aqueous solution.

As the particle approaches the wall, the fluid is squeezed out of the gap between the particle and the wall and a strong shear rate occurs in the gap region. One might argue that the pertinent length scale in this problem is the lubrication length scale as opposed to the particle diameter. From the continuity equation, the average radial velocity scales as $U_r \approx V_i\sqrt{ah}/h$ where h is the gap separation distance. Thus, the important shear rate in this problem is $\dot{\gamma} = U_r/h \approx V_i\sqrt{ah}/h^2$, which is based on the radial velocity and gap distance. The question is then: what is the appropriate h for this problem since h varies from $O(10a)$ to $O(10^{-5}a)$.

In the problem of particle-wall collision in liquids, the coefficient of restitution is defined as the ratio of the particle post-collision velocity at a specific position to the particle pre-collision velocity at the same position (h_0). For dry collisions, the coefficient of restitution does not depend on h_0 since after the particle separates from the wall, the change in particle velocity is negligible. However, for particle-wall collision in a viscous fluid, even after the particle separates from the wall, there is a marked decrease in particle velocity due to large viscous dissipation. See, for example, figure 14 in Ardekani & Rangel (2008). Here we define the coefficient of restitution at $h_0 = \delta t V_i$ where δt is the time between two photo frames. Figure 2(*d*) shows the graph obtained using the above definition for local strain rate ($\dot{\gamma} = \sqrt{a/\delta t^3 V_i}$). The data collapses onto the Newtonian data which is in agreement with the theoretical calculations for the Squeezing film of a shear-thinning fluid as explained above. We should clarify that this does not imply the dependence of the results on the camera speed. If we use a camera with higher speed but calculate the coefficient of restitution at the same h_0, we get the same data points. The definition of coefficient of restitution depends on h_0 but not the camera frame rate and it can be shown that its dependence on h_0 is smaller than the experimental uncertainty. If one changes the ratio of h_0 to surface roughness from 100 to 1000 using equation (4.4) of Joseph *et al.* (2001), the changes in coefficient of restitution will be smaller than the experimental uncertainty. In summary, the results for the coefficient of restitution in polymeric liquids collapse on the Newtonian fluid behaviour if one defines the Stokes number based on the local strain rate.

4. Conclusions

Particle-wall collision in a viscolastic liquid has been experimentally studied and the effects of both viscoelasticity and shear-thinning on the rebound velocity are discussed. It has been observed that the slope at which the coefficient of restitution increases with Stokes number is smaller for higher Deborah numbers. In addition, the critical Stokes number decreases with Deborah number. Higher rebound occurs for higher PEO concentration at the same stokes number. However, the results for the coefficient of restitution in polymeric liquids can be collapsed together with the Newtonian fluid behaviour if one defines the Stokes number based on the local strain rate (radial velocity of the fluid in the gap region divided by gap separation distance).

We wish to thank J. Garman for his help in setting up the experiment and S. Maghzi who measured surface roughness. This work is sponsored by the National Science Foundation under grants CBET-0302837 and OISE-0530270. The first author acknowledges the Zonta International Foundation for an Amelia Earhart fellowship.

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