Correction of Lamb's dissipation calculation for the effects of viscosity on capillary-gravity waves

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Purely irrotational theories of the flow of a viscous liquid are applied to model the effect of viscosity on the decay and oscillation of capillary-gravity waves. In particular, the dissipation approximation used in this analysis gives rise to a viscous correction of the frequency of the oscillations which was not obtained by Lamb's [H. Lamb, *Hydrodynamics* (Cambridge University Press, Cambridge, UK, 1932) (reprinted in 1993)] dissipation calculation. Moreover, our dissipation method goes beyond Lamb's in the sense that it yields an eigenvalue relation valid for the entire continuous spectrum of wave numbers. Comparisons are presented between the purely irrotational theories and Lamb's exact solution, showing good to reasonable agreement for long, progressive waves and for short, standing waves, even for very viscous liquids. The performance of the irrotational approximations deteriorates within an interval of wave numbers containing the cutoff where traveling waves become standing ones. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2760244]

I. INTRODUCTION

Stokes[1] introduced the idea of the dissipation method "in which the decay of the energy of the wave is computed from the viscous dissipation integral where the dissipation is evaluated on potential flow."[2] This method was implemented by Lamb[3] (Sec. 348) to study the effect of viscosity on the dynamics of free oscillatory waves on deep liquid. The waves are considered small departures about a plane-free surface. The result of his analysis was an estimate of the decay rate of the traveling waves. He also conducted the solution of the linearized Navier–Stokes equations for this problem using normal modes (Sec. 349; hereinafter "exact solution"), in which the zero-shear-stress condition at the free surface is satisfied. Independently, Basset[4] obtained the same dispersion relation as Lamb for the exact solution. Furthermore, Lamb[3] applied his dissipation method to study viscous effects on small oscillations about the spherical shape of a liquid globule in a vacuum or a bubble surrounded by liquid (Sec. 355). Whereas the effect of viscosity in both the decay rate and frequency of the oscillations can be examined through the exact solution, Lamb's dissipation approximation does not give rise to a viscous correction of the frequency.

In this work, we carry out the integration of the mechanical energy equation assuming irrotational flow to obtain a relation for the effects of viscosity on the decay rate and frequency of the oscillations of small capillary-gravity waves. Viscosity is explicitly considered in the dissipation term of the mechanical energy balance and the shear stress is set to zero at the free surface. This purely irrotational formulation is referred to as the dissipation method (DM) here. Our irrotational method is similar to Lamb's[3] in the sense that it is an irrotational approximation, but, unlike Lamb, we do not assume that the potential energy equals the kinetic energy and gravity is thus explicitly considered in our formulation. Unlike Lamb, our method of calculation yields a complex eigenvalue for progressive waves with the same growth rate as Lamb's but a different frequency that depends on viscosity. Lamb[3] advertises his method as valid to estimate "the effect of viscosity on free oscillatory waves on deep water." However, his analysis did not yield viscous effects on the frequency of these waves. In fact, when the dissipation method is carried out as presented here, it gives rise to progressive and standing waves just like the exact solution. The progressive waves are associated with long waves and the standing waves with short waves where the cutoff wave number is a decreasing function of the viscosity. For standing waves, DM predicts the effects of surface tension and gravity on the decay rate. Another purely irrotational theory of the motion of a viscous liquid is used in this study, namely the theory of viscous potential flow (VPF). In this approach, the viscous normal stress at the free surface enters the potential flow analysis.

Joseph and Wang[5] applied both VPF and the viscous correction of VPF (labeled as VCVPF) theories to the problem of free gravity waves in which capillary effects are neglected. In the latter approach, VPF theory is modified by adding a viscous pressure correction to the irrotational pressure to compensate the difference between the nonzero irrotational shear stress and the zero-shear stress at the free surface. Viscous effects in both the decay rate and the frequency were considered. The same decay rate obtained by Lamb's dissipation method was found from VCVPF. Wang and Joseph[6] performed a thorough comparison showing good agreement between the viscous irrotational theories, VPF and VCVPF, with Lamb's exact solution for short and long gravity waves, respectively, even for liquids with viscosity 10^4 times that of water. The theories of irrotational flow of a viscous fluid (VPF, VCVPF, and DM) are applied by Wang, Joseph, and Funada[7,8] to the problem of capillary instability and by Padrino, Funada, and Joseph[9] to study capillary-driven oscillations about the spherical shape of a drop or bubble. In these works, VCVPF and DM produced equivalent results. Although the study of nonlinear waves is beyond the scope of this paper, we mention that among the abundant literature on nonlinear waves, we know only the work by Longuet-Higgins[10] on the viscous effects on these waves. He computed the viscous decay of steep irrotational capillary-gravity waves through the dissipation method (see also Joseph, Funada, and Wang[11]).

Here, we compare and discuss predictions from the purely irrotational theories with the exact solution of the linearized problem. The method of calculating the dissipation applied in this work, which follows the steps of that used in the study of capillary instability,[7,8] leads to an excellent to reasonable approximation to the dispersion relation of the exact solution for long waves. For short waves, VPF gives the best approximation. The resulting performance may be cautiously used as a guide for application of the irrotational approximations in cases in which the "exact solution" is not known.

Neither Ref. 5 nor Ref. 6 applied the DM to waves on a plane free surface, although the former did review Lamb's dissipation approximation. The VPF calculation presented in this paper in Sec. II expands the procedure outlined in Ref. 5 to include surface tension effects. More importantly, it gives rise to intermediate results that are required by the DM in Sec. III. To the best of our knowledge, for the first time, viscous effects in both the decay rate and frequency of oscillations of "small" capillary-gravity waves about a plane free surface are obtained through the dissipation approximation.

II. VISCOUS POTENTIAL FLOW ANALYSIS (VPF)

Consider two-dimensional small irrotational disturbances of the basic state of rest of an incompressible fluid occupying half of the space, $-\infty \leq y \leq 0$, where y is a Cartesian coordinate such that the plane $y=0$ corresponds to the free surface for the basic state. The fluid in the upper half is dynamically inactive. The basic state is given by $dP/dy = -\rho g$, where P is the pressure, ρ is the liquid density, and g is the acceleration of gravity. We set, with no loss of generality, $P=0$ at the free surface of the undisturbed state $y=0$.

In the perturbed state, we look for functions that are periodic in the Cartesian coordinate x with period λ. The perturbed free surface has elevation $y=\eta(x,t)$. For irrotational flow, the velocity field is $\mathbf{u}=\nabla\phi$, such that the incompressibility condition implies Laplace's equation $\nabla^2\phi=0$ for the velocity potential ϕ.

Dynamical effects enter the analysis through the Bernoulli equation, which can be written as

$$\rho\frac{\partial\phi}{\partial t} + \frac{\rho}{2}|\nabla\phi|^2 + \hat{p} + \rho g y = 0, \quad \text{at } y=\eta, \tag{2.1}$$

where $\hat{p}$ is the pressure in the disturbed state. The balance of the normal stress at the free surface yields

$$-\hat{p} + 2\mu\frac{\partial^2\phi}{\partial n^2} = -\gamma\nabla_{\mathrm{II}}\cdot\mathbf{n} \quad \text{at } y=\eta, \tag{2.2}$$

where $\nabla_{\mathrm{II}}=\nabla-\mathbf{n}(\mathbf{n}\cdot\nabla)$.[12] In this expression, μ is the liquid dynamic viscosity and γ is the surface tension; the symbol $\mathbf{n}$ denotes the unit outward normal vector from the fluid at the free surface, which determines the n direction, and the second term on the left-hand side of (2.2) accounts for the viscous normal stress at the interface. At the free surface, the kinematic condition can be written as

$$\frac{\partial\phi}{\partial y} = \frac{D\eta}{Dt} = \frac{\partial\eta}{\partial t} + \mathbf{u}\cdot\nabla\eta \quad \text{at } y=\eta. \tag{2.3}$$

The pressure in the disturbed state is now decomposed into small disturbances about the basic state, such that $\hat{p}=P+p$ with $P=-\rho g y$. The system of equations (2.1)–(2.3) is linearized assuming that the free-surface displacement is "small" in comparison with the wavelength and has "small" slopes, $\partial\eta/\partial x \ll 1$. This process yields, at $y=0$,

$$\rho\frac{\partial\phi}{\partial t} + p = 0, \tag{2.4}$$

$$-p + \rho g\eta + 2\mu\frac{\partial^2\phi}{\partial y^2} = \gamma\frac{\partial^2\eta}{\partial x^2}, \tag{2.5}$$

$$\frac{\partial\phi}{\partial y} = \frac{\partial\eta}{\partial t}. \tag{2.6}$$

Solutions of Laplace's equation for ϕ in the domain $0\leq x\leq\lambda$, $-\infty\leq y\leq 0$ with periodic boundary conditions for $x=0$ and $x=\lambda$ can be written under the form

$$\phi = A e^{\sigma t + ikx + ky} + \text{c.c.}, \tag{2.7}$$

with $k=2\pi j/\lambda$ and $j=1,2,3,\ldots$, and c.c. denotes the complex conjugate of the previous term. The time dependence has been separated to obtain normal-mode solutions. We assume that the shape of the disturbed interface is also given by a normal-mode expression

$$\eta = \eta_0 e^{\sigma t + ikx} + \text{c.c.}, \tag{2.8}$$

where η_0 is a constant. Combining (2.4) and (2.5) to eliminate the pressure disturbance and using (2.7) and (2.8) gives rise to the expression

$$(\rho\sigma A + 2\mu A k^2 + \rho g\eta_0 + \gamma k^2\eta_0)e^{\sigma t + ikx} + \text{c.c.} = 0 \quad \text{at } y=0. \tag{2.9}$$

The linearized kinematic condition (2.6) is then used to find $Ak=\sigma\eta_0$. Applying this relation to eliminate η_0 in (2.9) yields the dispersion relation for VPF,

$$\sigma^2 + 2\nu k^2\sigma + gk + \gamma' k^3 = 0, \tag{2.10}$$

which can be solved for the eigenvalues

$$\sigma = -\nu k^2 \pm \sqrt{\nu^2 k^4 - (gk + \gamma' k^3)}, \tag{2.11}$$

with $\gamma'=\gamma/\rho$ and the kinematic viscosity $\nu=\mu/\rho$. This result can be obtained as a special case from the more general expression derived by Funada and Joseph[13] from the Kelvin-Helmholtz stability analysis of two viscous fluids. For $\gamma=0$ in (2.10), our result agrees with the eigenvalue relation presented by Joseph and Wang[5] using the theory of VPF for free gravity waves with no capillary effects.

In the case of $\nu k^2 \geq \sqrt{gk+\gamma' k^3}$ we have two real roots from (2.11) and the waves decay monotonically. The highest value gives the slowest decay rate. In the case of $\nu k^2 < \sqrt{gk+\gamma' k^3}$ we obtain the complex conjugate pair of roots

$$\sigma = -\nu k^2 \pm \mathrm{i}k\sqrt{(g/k+\gamma' k)-\nu^2 k^2}, \tag{2.12}$$

giving rise to progressive decaying waves. The decay rate is $-\nu k^2$, which is half of the value computed by Lamb using energy dissipation arguments. The speed of the traveling waves is

$$c = \sqrt{(g/k+\gamma' k)-\nu^2 k^2}, \tag{2.13}$$

and the speed is slower than the inviscid result $\sqrt{g/k+\gamma' k}$, as noticed by Joseph and Wang[5] in the absence of surface tension.

For short waves or high viscosity, i.e., $\nu k^2 \gg \sqrt{gk+\gamma' k^3}$, the eigenvalues from (2.11) follow

$$\sigma = -2\nu k^2 \quad \text{and} \quad \sigma = -\frac{\gamma' k}{2\nu} - \frac{g}{2\nu k}, \tag{2.14}$$

and the latter gives the slowest decay rate for the standing waves. For zero surface tension, the damping of the waves follows the rate $-g/(2\nu k)$, which decreases for shorter waves. This decay rate was found by Lamb[3] from the exact solution for large viscosity. For $\nu k^2 \ll \sqrt{gk+\gamma' k^3}$, traveling decaying waves are obtained and the eigenvalues behave as

$$\sigma = -\nu k^2 \pm \mathrm{i}k\sqrt{g/k+\gamma' k}, \tag{2.15}$$

such that the wave speed reaches the inviscid result.

III. DISSIPATION METHOD (DM)

The dissipation method relies on the integration of the mechanical energy equation. To apply DM to capillary-gravity waves, the working equation is obtained after subtracting the basic state of rest $\nabla P = \rho \mathbf{g}$ from the incompressible Navier-Stokes equation and then taking the scalar ("dot") product with the velocity vector. Integration over the region of interest yields the mechanical energy balance for the flow disturbances in integral form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \frac{\rho}{2}|\mathbf{u}|^2 \mathrm{d}V = \int_S \mathbf{n}\cdot\mathbf{T}\cdot\mathbf{u}\,\mathrm{d}S - \int_V 2\mu \mathbf{D}:\mathbf{D}\,\mathrm{d}V, \tag{3.1}$$

where $\mathbf{T}$ is the stress tensor for an incompressible Newtonian fluid in terms of pressure and velocity disturbances [see (3.2) below]; $\mathbf{D}$ is the strain-rate tensor

$$\mathbf{D} = \tfrac{1}{2}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T],$$

where the superscript T denotes the transpose, and $\mathbf{n}$ is the outward normal vector from the fluid. The symbol V denotes the volume of integration enclosed by the surface S. The last term in (3.1) gives the viscous dissipation. The double contracted product $\mathbf{D}:\mathbf{D} = D_{ij}D_{ji}$, and the repeated indexes imply summation according to Cartesian index notation in two dimensions.

The region of integration is defined by $0 \leq x \leq \lambda$ and $-\infty < y \leq 0$. Periodic boundary conditions at $x=0$ and $x=\lambda$ and disturbances (both velocity and pressure) that vanish as $y \to -\infty$ are considered. Therefore, the surface integral is reduced to an integral at $y=0$.

The first integral on the right-hand side of (3.1) can be expanded by considering

$$\mathbf{n}\cdot\mathbf{T}\cdot\mathbf{u} = (-p+\tau_{yy})v + \tau_{xy}u. \tag{3.2}$$

The analysis follows with the assumption that the zero-shear-stress condition and the normal-stress balance are satisfied at the free surface. Therefore, we have, at $y=0$,

$$\tau_{xy} = 0 \quad \text{and} \quad -p+\tau_{yy} = -\rho g \eta + \gamma \frac{\partial^2 \eta}{\partial x^2}. \tag{3.3}$$

Integrals in (3.1) are computed assuming that the fluid motion can be approximated as irrotational. The discontinuity of the zero shear stress at the free surface with the irrotational shear stress is resolved in a vorticity layer that is neglected in the analysis. For irrotational flow, the following identity holds:

$$\int_V 2\mu \mathbf{D}:\mathbf{D}\,\mathrm{d}V = \int_S \mathbf{n}\cdot 2\mu\mathbf{D}\cdot\mathbf{u}\,\mathrm{d}S. \tag{3.4}$$

Substitution of (3.2) and (3.4) into (3.1), using (3.3), yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \frac{\rho}{2}|\mathbf{u}|^2\mathrm{d}V = -\int_0^\lambda \rho g \eta v\,\mathrm{d}x + \int_0^\lambda \gamma \frac{\partial^2\eta}{\partial x^2} v\,\mathrm{d}x - \int_S \mathbf{n}\cdot 2\mu\mathbf{D}\cdot\mathbf{u}\,\mathrm{d}S. \tag{3.5}$$

Next, the integrals in (3.5) are carried out with the aid of the formula in the Appendix A and using expressions (2.7) and (2.8) for ϕ and η together with the relation $Ak = \sigma\eta_0$, which stems from the irrotational assumption. With $|\mathbf{u}|^2 = u^2 + v^2$ and writing the components of the strain-rate tensor $\mathbf{D}$ in Cartesian coordinates, such that $\mathbf{n}\cdot 2\mu\mathbf{D}\cdot\mathbf{u} = \tau_{yy}v + \tau_{xy}u$ from potential flow, this series of integrals gives rise to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \frac{\rho}{2}|\mathbf{u}|^2\mathrm{d}V = \rho k A\bar{A}(\sigma+\bar{\sigma})\mathrm{e}^{(\sigma+\bar{\sigma})t}\lambda, \tag{3.6}$$

$$\int_0^\lambda \rho g \eta v\,\mathrm{d}x = \rho g k^2 A\bar{A}\left(\frac{1}{\sigma}+\frac{1}{\bar{\sigma}}\right)\mathrm{e}^{(\sigma+\bar{\sigma})t}\lambda, \tag{3.7}$$

$$\int_0^\lambda \gamma\frac{\partial^2\eta}{\partial x^2}v\,\mathrm{d}x = -\gamma k^4 A\bar{A}\left(\frac{1}{\sigma}+\frac{1}{\bar{\sigma}}\right)\mathrm{e}^{(\sigma+\bar{\sigma})t}\lambda, \tag{3.8}$$

$$\int_S \mathbf{n}\cdot 2\mu\mathbf{D}\cdot\mathbf{u}\,\mathrm{d}S = 8\mu k^3 A\bar{A}\mathrm{e}^{(\sigma+\bar{\sigma})t}\lambda. \tag{3.9}$$

Substitution of (3.6)–(3.9) into (3.5) yields the expression

$$\sigma + 4\nu k^2 + (gk+\gamma' k^3)\frac{1}{\sigma} + \text{c.c.} = 0, \tag{3.10}$$

which is satisfied if the following eigenvalue relation holds:

$$\sigma^2 + 4\nu k^2\sigma + gk + \gamma' k^3 = 0 \tag{3.11}$$

with roots

$$\sigma = -2\nu k^2 \pm \sqrt{4\nu^2 k^4 - (gk + \gamma' k^3)}. \tag{3.12}$$

Putting $\gamma'=0$ in (3.12) yields the same relation obtained in Ref. 5 using the viscous correction of VPF, a method that follows a different path to the one described here. We regard (3.12) as an irrotational approximation for the exact solution.

For $2\nu k^2 \geq \sqrt{gk+\gamma' k^3}$, expression (3.12) gives two real roots and monotonically decaying waves are obtained. On the contrary, if $2\nu k^2 < \sqrt{gk+\gamma' k^3}$, progressive decaying waves occur. In this case, it is convenient to write (3.12) in the form

$$\sigma = -2\nu k^2 \pm \mathrm{i}k\sqrt{(g/k + \gamma' k) - 4\nu^2 k^2}, \tag{3.13}$$

and the traveling waves decay with rate $-2\nu k^2$, which is the same value obtained by Lamb[3] via the dissipation method. However, Lamb's approach did not account for the effects of viscosity in the wave speed of traveling decaying waves. The wave speed is extracted from (3.13) as

$$c = \sqrt{(g/k + \gamma' k) - 4\nu^2 k^2}, \tag{3.14}$$

which is slower than the inviscid result $\sqrt{g/k+\gamma' k}$. Prosperetti[14] finds (3.13) for small times and an irrotational initial condition from the solution of the initial-value problem for standing, capillary-gravity waves. He notes that this solution can apply to large viscosity. He also obtains Lamb's normal-mode solution (labeled "exact solution" here) as the asymptotic limit for large times, pointing out that the validity of Lamb's solution for all times is restricted to small viscosity.

For $2\nu k^2 \gg \sqrt{gk+\gamma' k^3}$ (e.g., short waves or high viscosity), relation (3.12) yields the trend for the decay rates

$$\sigma = -4\nu k^2 \quad \text{and} \quad \sigma = -\frac{\gamma' k}{4\nu} - \frac{g}{4\nu k}, \tag{3.15}$$

and the second root, which is governed by surface tension, gives the slowest decay rate of the standing waves. If $2\nu k^2 \ll \sqrt{gk+\gamma' k^3}$, as for long waves or low viscosity, the eigenvalues from (3.13) behave as

$$\sigma = -2\nu k^2 \pm \mathrm{i}k\sqrt{g/k + \gamma' k} \tag{3.16}$$

and the progressive decaying waves travel with the inviscid speed. Lamb[3] found that the exact solution reaches (3.16) in the case of "small" viscosity.

IV. DISCUSSION

Lamb[3] (Sec. 349) analyzed the viscous problem of small waves on the free surface of a deep liquid with capillary and gravity effects. The approach considers the solution of the linearized Navier-Stokes equations of an incompressible flow where the zero shear stress condition is satisfied at the free surface. Hence, vorticity is not set to zero. Lamb obtained the following eigenvalue relation, designated here as the exact solution:

TABLE I. Properties of the liquids used in this study.

Property	Water	Glycerin	SO10000 oil
ν ($m^2\,s^{-1}$)	1.00×10^{-6}	6.21×10^{-4}	1.03×10^{-2}
ρ ($kg\,m^{-3}$)	1.00×10^{3}	1.26×10^{3}	9.69×10^{2}
γ ($N\,m^{-1}$)	7.28×10^{-2}	6.34×10^{-2}	2.10×10^{-2}

$$[(\sigma + 2\nu k^2)^2 + \sigma_0^2]^2 = 16\nu^3 k^6(\sigma + \nu k^2), \quad \sigma_0^2 = gk + \gamma' k^3, \tag{4.1}$$

with $\mathrm{Re}[(\sigma+2\nu k^2)^2+\sigma_0^2] \geq 0$.

In this section, we compare the predictions from the exact solution with results from the purely irrotational theories, namely VPF, DM, and IPF, the inviscid irrotational theory. The latter is reached by setting $\nu=0$ in either VPF or DM eigenvalue relations (2.12) and (3.13), respectively; hence, the eigenvalues are purely imaginary with zero decay rate.

The dispersion relations (2.12), (3.13), and (4.1) can be conveniently written in dimensionless form as follows:

$$\text{VPF} \quad \tilde{\sigma} = -\theta \pm \mathrm{i}\sqrt{1-\theta^2}, \tag{4.2}$$

$$\text{DM} \quad \tilde{\sigma} = -2\theta \pm \mathrm{i}\sqrt{1-4\theta^2}, \tag{4.3}$$

$$\text{Exact solution} \quad [(\tilde{\sigma} + 2\theta)^2 + 1]^2 = 16\theta^3(\tilde{\sigma} + \theta), \tag{4.4}$$

and $\tilde{\sigma}=i$ for IPF. In these expressions, we have set $\tilde{\sigma}=\sigma/\sigma_0$ and $\theta=\nu k^2/\sigma_0$, a factor introduced by Lamb in his "exact solution." The analysis of (4.2)–(4.4) reveals that a threshold θ_c can be obtained that separates progressive waves ($\theta<\theta_c$, $\mathrm{Im}[\tilde{\sigma}]\neq 0$) from standing waves ($\theta \geq \theta_c$, $\mathrm{Im}[\tilde{\sigma}]=0$) from each theory. We obtain $\theta_c=1$ for VPF and $\theta_c=0.5$ for DM. For the exact solution, (4.4) gives rise to, nearly, $\theta_c=1.3115$ (also reported in Ref. 14). We notice that the first-order approximation in θ of (4.3) for the dissipation method is equivalent to the first-order approximation in this parameter of the exact solution presented by Lamb[3] and Basset.[4] As noted by Landau and Lifshitz[15], the vortical layer is thin when viscous effects are "weak" (small θ) and its contribution to the total energy dissipation is negligible in comparison to the dissipation from within the bulk of fluid.

From the definition of θ, we have that the respective cutoff wave number k_c can be obtained for each theory using the corresponding value of θ_c given above. When $k<k_c$, progressive waves decay, whereas for $k \geq k_c$, the waves decay monotonically.

To investigate the crossover from progressive to standing waves according to the exact solution and the irrotational approximations, we choose three different liquids, namely a highly mobile one such as water, glycerin, and SO10000, a very viscous oil at ambient temperature. The properties of these liquids used in the computations are indicated in Table I. The kinematic viscosity varies several orders of magnitude from one fluid to another. The cutoff wave number k_c from the exact solution, VPF, and DM is shown in Table II for the three liquids. For the same liquid, DM gives the lowest k_c and the exact solution gives the largest. Therefore, traveling waves of certain length according to the exact solution may

TABLE II. Cutoff wave number k_c (m^{-1}) computed for DM, VPF, and the exact solution for three different liquids: water, glycerin, and SO10000 oil.

Theory	Water	Glycerin	SO10000 oil
DM	1.82×10^7	196.81	28.50
VPF	7.28×10^7	344.64	45.29
Exact	1.25×10^8	445.18	54.30

be predicted as standing waves by the irrotational approximations. Table II reveals that the cutoff wave number decreases as the viscosity increases. Thus, the region of progressive decaying waves $0<k<k_c$ shrinks with increasing viscosity. Waves for which $k<k_c$ oscillate with a finite period whereas waves with $k>k_c$ can be though of as having an infinitely long period. This latter feature is clearly associated with the low mobility of highly viscous liquids.

Figure 1 shows the dimensionless decay rate $-\text{Re}[\tilde{\sigma}]$ and frequency of the oscillations $\text{Im}[\tilde{\sigma}]$ as a function of the dimensionless parameter θ from (4.2)–(4.4) for VPF, DM, and the exact solution, respectively. IPF predictions for the frequency, $\text{Im}[\tilde{\sigma}]=1$, are also included. For $\theta>\theta_c$ only the slowest decay rate, given by the smallest real eigenvalue, is plotted. In these figures, only the cutoff θ_c given by the exact solution is presented, which hereafter is referred to as θ_c'. An important feature of the dimensionless representation of the dispersion relations (4.2)–(4.4) is that their graphs of $\tilde{\sigma}$ versus θ are equally applicable to any incompressible Newtonian fluid, and no individual plots have to be presented for every liquid chosen.

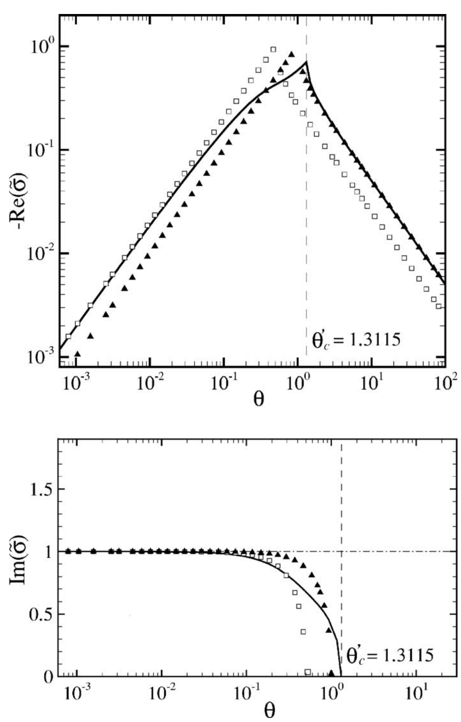

FIG. 1. Dimensionless decay rate $-\text{Re}[\tilde{\sigma}]$ and frequency of the oscillations $\text{Im}[\tilde{\sigma}]$ as a function of the dimensionless parameter $\theta=\nu k^2/\sigma_0$ from the exact solution and the irrotational theories VPF, DM, and IPF: Solid line, exact solution; ▲, VPF; □, DM; dashed-dotted line, IPF. In the latter case, the eigenvalues $\tilde{\sigma}$ are purely imaginary. The dimensionless eigenvalue is $\tilde{\sigma}=\sigma/\sigma_0$, where σ_0 is the inviscid frequency. For $\theta>\theta_c$, the solutions are purely real; in this case, only the slowest decay rate, given by the lowest real eigenvalue, is presented. The cutoff θ_c' corresponds to the exact solution.

Both viscous irrotational theories follow the trend described by the exact solution as shown in Fig. 1. With respect to the decay rate, this figure indicates that DM approaches the exact solution in the progressive-wave regime $(\theta<\theta_c')$ for $\theta\ll1$. In particular, for $\theta\leq0.02$ we found that the relative error for DM in absolute value remains below 10% and the agreement becomes outstanding as θ decreases since $\text{Re}[\tilde{\sigma}]=-2\theta$ as indicated by (3.16). On the other hand, VPF is off the mark by 50%, as can be anticipated from (2.15). In the standing-wave regime $(\theta\geq\theta_c')$, VPF shows excellent agreement with the exact solution; for $\theta\geq2$, this irrotational theory predicts values of the decay rate with relative errors within 5% in absolute value and the agreement with the exact solution improves substantially following $\tilde{\sigma}=-1/(2\theta)$ as θ increases as predicted by (2.14). By contrast, DM underpredicts the decay rate by 50% in this regime in accord with (3.15). In the transition region $(0.02\leq\theta\leq2$, say), neither of the viscous irrotational theories gives a good approximation of the exact decay rate for this entire interval. However, each of these approximations gives rise to a critical value θ_c that qualitatively resembles the crossover from progressive to standing waves depicted by the exact solution.

Regarding the frequency of the oscillations, $\text{Im}[\tilde{\sigma}]$, Fig. 1 reveals that viscous effects are significant when $\theta\geq0.1$, for which the exact solution deviates from the inviscid result. The frequency becomes damped and, for $\theta\geq\theta_c'$, the oscillations are suppressed. These features in the dynamics of the waves are also described, on qualitative grounds, by the viscous irrotational approximations. This figure also illustrates what was computed above, namely the lowest crossover θ_c is given by DM and the highest is obtained from the exact solution; the cutoff from VPF lies in between. This cutoff between progressive and standing waves cannot be obtained from the dissipation calculation implemented by Lamb based on Stokes' idea.

An aspect that is worth mentioning and which may not be evident from the graph $-\text{Re}(\tilde{\sigma})$ versus θ is the effect of surface tension in the decay of the waves. The slowest decay rate for the exact solution and VPF goes as $-\gamma'k/(2\nu)$ as $k\rightarrow\infty$, whereas for DM, it goes as $-\gamma'k/(4\nu)$. Thus, as the waves become shorter, they are more rapidly damped by capillary effects. By contrast, the suppression of the regularizing effect of surface tension yields a decrease in the decay rate of the gravity waves as k increases, as shown by Wang and Joseph[6] [see (2.14) and (3.15)]. In the case of the inviscid theory, the frequency continues increasing as $k\rightarrow\infty$ and the waves oscillate undamped. The shortest the wave, the highest the frequency, which contradicts the viscous theories. By taking into account viscous effects in the irrotational theories, the crossover from the traveling-wave regime to the standing-wave regime is predicted by these approximations, in a similar fashion as the exact solution.

In this analysis, we have shown that the effect of viscosity on the frequency of capillary-gravity waves, which Lamb[3] assumed to be the same as in an inviscid fluid, can be obtained from the dissipation integral in the mechanical energy balance. Moreover, results from this dissipation method are not restricted to oscillatory waves, as is the case with Lamb's dissipation calculation, but they also predict values for the decay rate of standing waves that follow the trend described by the exact solution. In sum, our dissipation method yields an eigenvalue relation for the entire spectrum of wave numbers and is in good agreement with the exact solution for sufficiently large waves. VPF is the best approximation for sufficiently short waves.

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APPENDIX: INTEGRATION FORMULA

In the analysis presented in Secs. III and IV, the following formula is used:

$$\int_S (B+\bar{B})(C+\bar{C})\mathrm{d}S = 2\int_S \mathrm{Re}[BC + B\bar{C}]\mathrm{d}S = 2\,\mathrm{Re}\left[\int_S (BC+B\bar{C})\mathrm{d}S\right], \tag{A1}$$

where S denotes the region of integration and B and C are complex fields. The bar indicates complex conjugate and $\mathrm{Re}[\cdot]$ is a linear operator that returns the real part of a complex number.

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