Capillary Collapse and Rupture

T.S. Lundgren

University of Minnesota, Minneapolis, MN 55455

D.D. Joseph

University of Minnesota, Minneapolis, MN 55455

(June 9, 1998)

Abstract

The breakup of a liquid capillary filament is analyzed as a viscous potential flow near a stagnation point on the centerline of the filament towards which the surface collapses under the action of surface tension forces. We find that the neck is of parabolic shape and its radius collapses to zero in a finite time; the curvature at the throat tends to zero much faster than the radius, leading ultimately to a microthread of nearly uniform radius. During the collapse the tensile stress due to viscosity increases in value until at a certain finite radius, which is about 1.5 microns for water in air, the stress in the throat passes into tension, presumably inducing cavitation there.

INTRODUCTION

The breakup of liquid jets is generally framed in terms of the capillary pressure $\sigma/R(z,t)$ due to surface tension σ acting at the neck of radius $R(z,t)$. The capillary pressure is greatest at the position z where R is smallest, an unstable situation in which liquid is squeezed out of the neck further reducing R and increasing the capillary pressure there. This picture leads to an inevitable collapse of the radius to zero. The conventional view is that the capillary instability just described leads to "pinchoff" but the physics required to actually rupture the thread is not understood. Here we are promoting the idea that the filament ruptures by cavitation due to tensile stresses induced by the motion out of the neck. The idea that liquids can cavitate by tensile stresses associated with motions, rather than by lowering the pressure was introduced by Joseph [3]. One of the interesting implications of this idea [4] is that cavitation in a pure shear flow may be induced by a tensile stress at $45°$ from the direction of shearing.

In this paper we present a simple collapsing solution in which all the physics is transparent. The solution is a viscous potential flow. In most viscous potential flows applied to free surface problems, such as the celebrated Levich [5] rising bubble problem, continuity of the shear stress at the free surface goes unsatisfied, justified for large Reynolds numbers by the weakness of the thin shear layer which develops. In the solution presented here continuity of the shear stress is satisfied asymptotically at all Reynolds numbers.

ANALYSIS

Here we consider the collapse of a capillary filament under surface tension forces which squeezes liquid symmetrically from a parabolic throat sketched in figure 1. We assume that the flow in the neighborhood of the throat is an axially symmetric straining flow, or stagnation point flow, with velocity components

$$u_z = a(t)z \tag{1}$$

$$u_r = -\frac{1}{2}a(t)r \tag{2}$$

and determine the strain rate $a(t)$ and the capillary shape, $r = R(z,t)$, by satisfying the appropriate boundary conditions at the capillary surface. The velocity field described by equations (1) and (2) is incompressible and irrotational, therefore despite being a viscous flow, it may be described by a velocity potential ($u_z = \frac{\partial\varphi}{\partial z}, u_r = \frac{\partial\varphi}{\partial r}$) of form

$$\varphi = \frac{1}{2}az^2 - \frac{1}{4}ar^2. \tag{3}$$

The pressure in the flow is determined from the unsteady version of the Bernoulli equation

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}(u_r^2 + u_z^2) + \frac{p}{\rho} = \frac{p_0}{\rho} \tag{4}$$

in the form

$$\frac{p - p_0}{\rho} = -(\frac{1}{2}\dot{a} + \frac{1}{2}a^2)z^2 + (\frac{1}{4}\dot{a} - \frac{1}{8}a^2)r^2. \tag{5}$$

The stagnation pressure p_0 is generally an unknown function of time in this local solution. For sufficiently large Reynolds numbers equation (4) will be valid beyond the local region and p_0 will equal the *constant* pressure in a distant state of rest, providing a global reference in this otherwise local solution. The overdot denotes a time derivative.

In this flow the state of stress is given by two principal stresses;

$$T_{zz} = -p + 2\mu\frac{\partial u_z}{\partial z} = -p + 2\mu a, \tag{6}$$

$$T_{rr} = -p + 2\mu\frac{\partial u_r}{\partial r} = -p - \mu a. \tag{7}$$

The normal traction at a point on the free surface, the force per unit area which the surface exerts on the fluid, is

$$T_{nn} = n_r^2 T_{rr} + n_z^2 T_{zz} \tag{8}$$

where n_r and n_z are components of the unit outward normal (indicated in figure 1).

A force balance at the free surface gives the boundary condition

$$-T_{nn} - p_a = \sigma\kappa \tag{9}$$

where p_a is atmospheric pressure, σ is the surface tension force per unit length and κ is the mean curvature, given by

$$\kappa = -\frac{\frac{\partial^2 R}{\partial z^2}}{(1+(\frac{\partial R}{\partial z})^2)^{\frac{3}{2}}} + \frac{1}{R(1+(\frac{\partial R}{\partial z})^2)^{\frac{1}{2}}} \quad . \tag{10}$$

Equation (9) is the condition which drives the capillary collapse.

The condition of zero shear stress at the boundary is not exactly satisfied. It is well-known [10] that vorticity is generated at free surfaces: the condition that the shear stress is zero prevents the flow from being exactly irrotational. In this axially symmetric flow the vorticity is required to be $\omega_\theta = 3an_rn_z$, which can be regarded as a measure of the error. For large Reynolds numbers the vorticity generated is confined to a thin layer with thickness of order $\mathrm{Re}^{-1/2}$ with variation of velocity across it of order $\mathrm{Re}^{-1/2}$. That is, the vortex layer is thin and weak. However, even if the Reynolds number is not small the magnitude of the vorticity required is small for this flow, because n_z is very small in our solution.

Since the free surface must move with the fluid, we also have the kinematic condition

$$u_r = \frac{\partial R}{\partial t} + u_z \frac{\partial R}{\partial z} \tag{11}$$

at $r = R(z,t)$. This may be written

$$-\frac{1}{2}aR = \frac{\partial R}{\partial t} + az\frac{\partial R}{\partial z}. \tag{12}$$

The mathematical problem is to find a function $R(z,t)$ which satisfies the conditions expressed by equations (9) and (12). We will show that a function of form

$$R(z,t) = R_0(t) + R_2(t)z^2 + O(z^4) \tag{13}$$

is suitable and determine $R_0(t), R_2(t)$ and the strainrate $a(t)$ by expanding these conditions for small z. To the lowest order in z^2

$$\frac{T_{nn}}{\rho} = \frac{T_{rr}}{\rho} = -\frac{p}{\rho} - \nu a \tag{14}$$

$$= -\frac{p_0}{\rho} - (\frac{1}{4}\dot{a} - \frac{1}{8}a^2)R_0^2 - \nu a \tag{15}$$

and to the same order

$$\kappa = \frac{1}{R_0} - 2R_2, \tag{16}$$

so equation (9) gives

$$\frac{p_0 - p_a}{\rho} + (\frac{1}{4}\dot{a} - \frac{1}{8}a^2)R_0^2 + \nu a = \frac{\sigma}{\rho}(\frac{1}{R_0} - 2R_2) \quad . \tag{17}$$

Equation (12) gives the two equations

$$-\frac{1}{2}aR_0 = \dot{R}_0 \tag{18}$$

and

$$-\frac{5}{2}aR_2 = \dot{R}_2. \tag{19}$$

From these we see that $R_2 = CR_0^5$, where C is a constant depending on starting conditions. This result implies the R_2 tends to zero faster than R_0 which means the parabola flattens out during collapse. This flattening can be identified with the formation of a cylindrical "micro-thread" of nearly constant radius which appears in all the many different cases studied by Kowalewski [9]. In particular, Kowalewski's figure 10 shows the length of the micro-thread increasing with this before collapse.

When equation (18) is used to eliminate a from equation (17) we obtain a single equation for $R_0(t)$, namely,

$$\frac{p_0 - p_a}{\rho} - \frac{1}{2}R_0\ddot{R}_0 - \frac{2\nu\dot{R}_0}{R_0} = \frac{\sigma}{\rho}(\frac{1}{R_0} - 2CR_0^5). \tag{20}$$

This equation is structurally similar to the Rayleigh-Plesset equation [8] which describes the collapse and growth of spherical bubbles by a viscous point source potential flow. Since we are most interested in small R_0, we can pick out the dominant terms as R_0 tends to zero. These are

$$-\frac{2\nu\dot{R}_0}{R_0} = \frac{\sigma}{\rho R_0} \tag{21}$$

which is a balance between the *viscous* part of the normal force (which resists the collapse) and the surface tension force (which drives it). The large R_0^{-1} term cancels from each side giving

$$\dot{R}_0 = -\frac{1}{2}\frac{\sigma}{\rho\nu} \tag{22}$$

with solution

$$R_0 = \frac{\sigma}{2\rho\nu}(t_* - t) \tag{23}$$

where t_* is a constant of integration. Therefore we have a solution in which R_0 tends to zero in a finite time. It is easy to see that the neglected terms in equation (20) give a correction to R_0 of order $(t_* - t)^2$. With the additional term the solution becomes

$$R_0 = \frac{\sigma}{2\rho\nu}(t_* - t) - \frac{\sigma}{2\rho\nu}\frac{p_0 - p_a}{4\rho\nu}(t_* - t)^2 + \cdots, \tag{24}$$

The strain rate in the flow,

$$a = -2\frac{\dot{R}_0}{R_0} = \frac{2}{t_* - t} - \frac{p_0 - p_a}{2\rho\nu} + \cdots, \tag{25}$$

becomes infinite as

$t \to t_*$ and the pressure, given by equation (5) is

$$\frac{p}{\rho} = \frac{p_0}{\rho} - \frac{3z^2}{(t_* - t)^2} + \frac{(p_0 - p_a)}{\rho\nu}\frac{(z^2 + \frac{1}{4}r^2)}{t_* - t} + \cdots \tag{26}$$

to leading orders.

The result given by equation (24) says that R_0 tends to zero linearly in time. Eggers [2] presented a solution to model equations which also exhibits linear collapse. A *nonviscous* similarity law due to Keller and Miksis [6] predicts that the radius will collapse to zero more rapidly than this, like $(t_* - t)^{\frac{2}{3}}$.

Equation (26) shows that the pressure decreases away from the throat at $z = 0$ and would tend to zero a short distance away. This would lead to cavitation in the traditional sense. However the solution is limited to small values of z because of the power series expansion. In the analysis to follow, we show that cavitation can occur even at $z = 0$.

The stress component T_{zz} at the stagnation point $(z = 0, r = 0)$ is

$$\frac{T_{zz}}{\rho} = -\frac{p_0}{\rho} + 2\nu a \tag{27}$$

$$= -\frac{(2p_0 - p_a)}{\rho} + \frac{4\nu}{t_* - t} + O(t_* - t). \tag{28}$$

It is convenient to express this in terms of the instantaneous value of the throat radius by substituting $t_* - t = 2\rho\nu R_0/\sigma$ to get

$$\frac{T_{zz}}{\rho} = -\frac{(2p_0 - p_a)}{\rho} + \frac{\Omega\sigma}{\rho R_0(t)} \tag{29}$$

Equation (29) shows that the capillary thread will pass into tension when

$$R_0(t) < R_{ocr} = \frac{2\sigma}{(2p_0 - p_a)} \tag{30}$$

and the tensions will grow unboundedly large as $R_0(t) \to 0$. Liquids which are specially prepared to remove nucleation sites cannot withstand large tensions, and impure liquids like tap water cannot sustain any tension. The critical value R_{ocr} is independent of viscosity for low viscosity fluids since we can use a global value for R_0. If we estimate p_0 to be approximately $p_a = 10^6$dynes/cm^2 for water with $\sigma = 75$ dynes/cm we get $R_{ocr} = 1.5$ micrometers. For high viscosity fluids p_0 probably depends on viscosity.

The solution presented here may be compared with the experiments of Kowalewski [9]. The most striking features of these experiments is the appearance of a thin liquid neck of nearly constant radius, a micro-thread, joining the droplet to a much fatter macro-thread. This thread elongates and thins until it ruptures. "... Its final diameter before rupture was approximately one micrometer and seems constant within wide range of parameters varied." In comparing theory with these experiments, perhaps the most salient point of agreement is the vanishing of variations of curvature over an increasing length of the micro-thread.

The observed dynamics of the micro-thread is in astonishing agreement with simple extension of a cylinder of constant radius satisfying (1) and (2) and the agreements, surprisingly, are even better for the more viscous threads.

A Reynolds number for the collapsing capillary may be defined by

$$R_e = \frac{R_0 \dot{R}_0}{\nu}, \tag{31}$$

based on the throat radius and the velocity of collapse, using equation (22) for the latter quantity gives

$$R_e = \frac{R_0\sigma}{2\rho\nu^2}, \tag{32}$$

which is the ratio of R_0 to a viscous length [7] $2\rho\nu^2/\sigma$ which is very small for water, about .027 micron. Therefore using $R_{0\mathrm{cr}}$ (=1.5 micron) for R_0, the Reynolds number at collapse is about 55 for water (the collapse velocity is about 37 m/s). For more viscous liquids the Reynolds number at collapse could be very small.

The symmetric local solution derived here may not be stable; photographs of breaking liquid bridges [7] and jets [ ] are globally asymmetric. A strongly collapsing capillary could be expected to amplify asymmetries, as is known to happen in a collapsing bubble. However, thinning micro-threads of nearly constant radius which evolved from the pinched asymmetric macro-threads observed in the jet break-up experiments of Kowalewski [9] look exactly like thinning threads of vanishing axial curvature which we have computed. An extending, thinning thread of vanishing axial curvature can only be described as an extensional flow satisfying (1) and (2) with the caveat that the position of the stagnation point in the experiments is not known. In such a flow, the pressure will decrease away from the origin and the extensional stress (29) due to stretching will be uniform along the thread and will dominate for small threads. In this case thread rupture would be expected to occur at various and unpredictable nucleation sites as in the experiments of Kowalewski [9].

CONCLUSIONS

Neckdown of a liquid capillary thread was studied in a local analysis based on viscous potential flow. One objective of this study was to show that during collapse the thread will enter into tension due to viscosity and can be expected to fracture, or cavitate, at a finite radius.

The flow in the throat of the collapsing capillary is locally a uniaxial extensional flow, linear in z and r, with a time dependent strain rate $a(t)$. This viscous potential flow satisfies the Navier-Stokes equation and all the relevant interfacial conditions, including continuity of the shear stress.

The principal dynamic balance is between the surface tension forces, which are trying to collapse the capillary, and the radial viscous stress which is resisting the collapse. Since mass must be conserved a large axial flow results from squeezing liquid out of the neck and this results in a large viscous extensional stress. The extensional stress passes into tension at $R_0 = 1.5$ micron (for water and air) long before R_0 actually collapses to zero.

The solution is symmetric about $z = 0$, the position of the smallest radius; the axial velocity is odd and the radial velocity, pressure and interface shape

$$R(z,t) = R_0(t) + R_2(t)z^2 + O(z^4)$$

are even in z. At lowest order the interface is a parabola in which $R_2(t)$ is proportional to R_0^5, hence in the limit of collapsing radius $R_2 \to 0$ much more rapidly then R_0 and the shape approaches that of a straight cylinder. This solution, like the micro-threads observed by Kowalewski [9] have a *vanishing axial curvature*. Though they evolve from different configurations, the observed micro-threads and our theoretical solution both tend to a thinning of uniform but decreasing radius, pure extension of a thin cylinder. The radius tends to zero linearly, like $(t_* - t)$, collapsing to zero in a finite time. At the same time the strain rate $a(t)$ tends to infinity like $(t_* - t)^{-1}$.

Acknowledgment

The work of DDJ was supported by the Particulate Muliphase Processes division of the NSF under a grant for exploratory research, and ARO Mathematics.

REFERENCES

[1] M.P. Brenner, J. Eggers, K. Joseph, R. Nagel and X.D. Shi, Breakdown of scaling in droplet fission at high Reynolds number, *Phys. Fluids* **9**, 1573-1590 (1997).

[2] J. Eggers, Universal pinching of 3D axisymmetric free-surface flow, *Phys. Rev. E.*, **71**, 3458-3460, (1993).

[3] D.D. Joseph, Cavitation in a flowing liquid, *Phys. Rev. Lett.*, **51**, 1649-1650 (1995).

[4] D.D. Joseph, Cavitation and the state of stress in a flowing liquid, *under review, see: http://www.aem.umn.edu/ people/faculty/joseph/papers/download.html*; (1997).

[5] V.G. Levich, The motion of bubbles at high Reynolds numbers, *Zh. Eksp. Teor. Fiz,* **19**, 18 (1949).

[6] J. Keller and M.J Miksis, Surface tension driven flows, *SIAM J. Appl. Math,* **43**, 268 (1983).

[7] D.H. Peregrine, G. Shoker and A. Symon, The bifurcation of liquid bridges, *J. Fluid Mech.*, **212**, 25-39, (1990).

[8] M.S. Plesset and A. Prosperetti, Bubble dynamics and cavitation, *Ann. Rev. Fluid Mech.*, **9**, 145-185 (1977).

[9] T.A. Kowalewski, On the separation of droplets from a liquid jet, *Fluid Dyn. Res.*, **17**, 121-145, (1996).

[10] Batchelor, pp. 365, (1967).

FIGURES

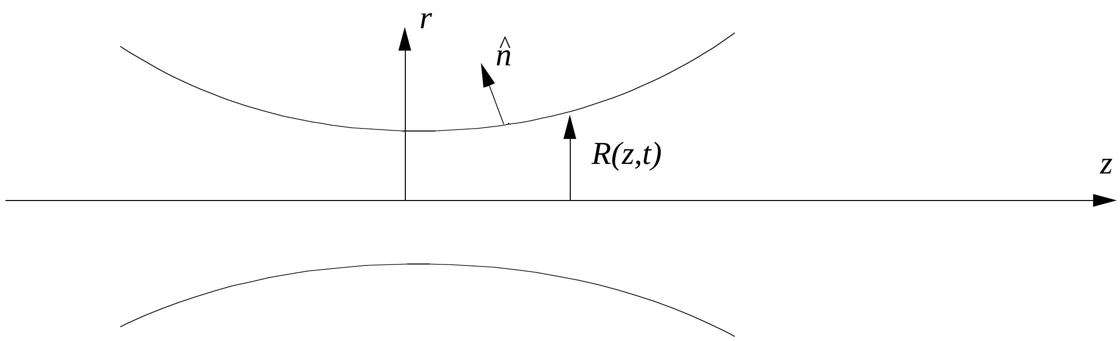

FIG. 1.