A Maxwell memory model for delayed weather response to solar heating

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Abstract

A linear Maxwell-type viscoelastic model, relating seasonal variations of temperature at any given place on the earth to variations in the length of the day, is proposed. Comparison with observations shows excellent agreement for mid-latitudes, and the two free parameters in the model (the memory parameter, or the phase lag, and the effective viscosity coefficient) are extracted. Simple formulae connecting these parameters are presented. An interpretation of the results, and some additional remarks on temperature variations at low and high latitudes, are provided.

1 Introduction

1.1 Motivation

The temperature experienced at any given place on earth is ultimately related to the solar radiation received by the earth and the (differential) storage and rerelease of this radiation by oceans and the land. Much detailed work has gone into the characterization of these parameters, and of the way in which they affect both the local weather and the long-term climate. We ignore such details in this paper, and model the observed seasonal temperature variations in a simple but effective way.

The model is motivated by the observation that the warmest day of the year in the northern hemisphere occurs, on the average, sometime in the latter half of July, and that the coldest day of the year occurs sometime in the latter half of January – each of them a little over a month after the respective solstice days. (It may be recalled that the distance between the sun and earth's equator is the greatest on the two solstice days, which correspond to the longest and the shortest days of the year.) The two days on which the rate of change of the average daily temperature is maximum also follow the vernal and autumnal equinox by about a month. In a somewhat simplistic way, one can interpret these observations to mean that the earth's temperature responds with memory to the input of energy from the sun, the relevant time lag being of the order of a month. This consideration suggests a memory model for temperature variations, which in its simplest version is the Maxwell model [1] involving two parameters, one being a relaxation time (or memory parameter) and the other an (effective) viscosity coefficient. The spirit of the model is that ocean-land interaction and other such details, which it entirely ignores, can be subsumed adequately by these two parameters. If the model has some merit to it, one should be able to determine these parameters and interpret them usefully. The purpose of this paper is to show that this is essentially the case.

1.2 The model

The simplest version of the model considers, at any latitude ℓ, the temperature averaged over all positions for that latitude; it further considers the mean temperature on any given day of the year averaged over many years, so that statistical fluctuations from one year to another are ignored. Let this average temperature be denoted by $\theta(\ell, t)$ where t denotes the number of days within a year, elapsed (for example) since the occurrence of one of the two solstices. Let $< \theta >$ be the time average of θ, taken over all days in a year, and T denote the difference $\theta - < \theta >$. At any latitude, we are interested in the variation of T through the year, and wish to relate it to $L = L^* - < L^* >$, where $L^*(t)$ is the length of the day at that latitude and $< L^* >$, the yearly average of L^*, is approximately 12 hours at all latitudes. The proposed model states that

$$\lambda \frac{\partial T}{\partial t} + T = \eta L(t). \quad (1)$$

where λ is a relaxation (memory) parameter and η is a viscosity coefficient. The model is linear.

It should be noted that, although models resembling equation (1) have been proposed for seasonal variations of temperature[2], they are generally nonlinear and more complex (incorporating, for example, ice-albedo feedback and turbulent eddy diffusvity). We know of no past effort identical to ours incorporating a time constant as well as a viscosity coefficient, and carrying out data analysis to this degree of completion.

2 Data analysis

2.1 The length of the day

Figure 1 shows a plot of $L(t)$ at 50 deg latitude in the northern hemisphere over one year period, plotted as a function of time counted since the occurrence of summer solstice. The data are available in tabulated form[3] for latitudes lower than 65 deg; the data become increasingly uncertain for higher latitudes because small changes in atmospheric refractivity can cause relatively large changes in daylight, as can small uncertainties in latitudes. This difficulty restricts our consideration here to latitudes below 65 deg. Figure 1 shows that $L(t)$ can be fitted closely enough by a single cosine term, $Acos(\omega t)$, where A is the amplitude of the daylight variation and ω is the circular frequency given by $(2\pi/365)\ days^{-1}$. The smooth variation of A with respect to the latitude can be fitted by a low-order polynomial: The simple fit given by

$$\ell = 19.47A - 2.10A^2 + 0.082A^3, \tag{2}$$

where the latitude ℓ is expressed in degrees and the amplitude of daylight variations A is in hours, seems to be quite adequate (Fig. 2).

2.2 Temperature variations in mid-latitudes

If $L(t)$ in equation (1) can be fitted by a cosine term, it is clear that $T(t)$ can be expressed as

$$T(t) = a\ cos\omega t + b\ sin\omega t. \tag{3}$$

where

$$\lambda = \frac{b}{a\omega} \tag{4}$$

and

$$\eta = \frac{a + b\lambda\omega}{A}. \tag{5}$$

We should now check if equation (3) adequately expresses the mean temperature variation through the year and, if so, obtain the constants a and b; using equations (4) and (5), one can then extract λ and η. This will be done below.

Figure 3 shows that, indeed, equation (3) adequately fits the temperature data at 50 deg latitude. Use of equations (4) and (5) yields, for this latitude,

$$\lambda = 38\ days,\ \ and\ \eta = 2.69\ ^{\circ}C/hr. \tag{6}$$

The value of the relaxation constant λ is roughly consistent with our expectation. Considering that η represents the increase in temperature per hour of heating by the sun, it appears that, on the average, an hour of solar heating will typically raise the temperature at 50 deg latitude by about $2.7^{o}C$.

In analysing temperature data for other latitudes, two restrictions should be noted. First, as already mentioned, for latitudes above 65 deg, no reliable data exist on the length of the day. Therefore we have not extensively examined the validity of the model for higher latitudes, although temperature variations up to a latitude of 75 deg (this being the limit of the data available to us) can indeed be fitted by equation (3). For latitudes below about 30 or 25 deg, the smaller yearly temperature variations become influenced by several effects, none of which (including the length of the day) appears to have a particularly predominent influence. This limits the applicability of the model. Possibilities for improvement for low latitudes will be suggested briefly later but, for now, we restrict attention chiefly to the latitude region between 30 deg and 65 deg (which shall henceforth be called mid-latitudes). We consider the northern hemisphere almost exclusively because the data for the southern hemisphere are rather sparse; however, the available data (such as they are) in the southern hemisphere can be handled in an identical way.

Our experience is that temperature variations at *all* mid-latitudes can be fitted well by equation (3). A quick feel for the goodness of this fit can be had from Fig. 4 which compares one-half of ΔT_{max}, the difference between the maximum and minimum temperatures at various mid-latitudes, with the amplitude $(a^2 + b^2)^{\frac{1}{2}}$ obtained from fitting the data to equation (3). The agreement is good. It may be useful to note that the data can be approximated by a straight line which intercepts the latitude axis at the finite latitude of about 13 deg. One can combine this approximate and empirical observation with equations (4) and (5) to write

$$\frac{\eta A}{\sqrt{1+\lambda^2\omega^2}} = \Delta T_{max} = (a^2 + b^2)^{\frac{1}{2}} \;=\; 0.24\,\ell - 1.8. \qquad (7)$$

Here, ΔT_{max} is expressed in ^oC and the latitude ℓ in degrees. Equation (7) relates all three parameters λ, η and A, and is a simple formula for ΔT_{max} as a function of latitude.

Figure 5 shows that both the relaxation time λ and the viscosity η decrease with increasing latitude, with the latter showing a stronger dependence. However, to within about 15%, the ratio $\frac{\lambda}{\eta}$ is a constant of about 15. The increasing value of λ with decreasing latitude was not expected at the outset. To assuage the skepticism of the reader who may be similarly skeptical, we plot in Fig. 6 the temperature variations for latitudes of 30 and 75 degrees. It is clear that the warmest day occurs sooner in the year at higher latitudes than at lower latitudes. The meaning of this observation will be discussed subsequently.

The data examined so far are for averages at a given latitude. Similar analysis can be done for local regions on the globe. Figure 7 shows the temperature variation through the year for the city of New Haven. Examination of the data for several other cities in American continent shows that equations (3)-(5) adequately describe the observed temperature variations although, not unexpectedly, λ does vary somewhat from one city to another even when the cities lie on comparable latitudes (Table 1). The general trend for λ to increase with decreasing latitudes is quite clear.

2.3 Temperature variations at different heights

We have examined the temperature data at several altitudes from the ground. There are only minor variations with respect to height, at least until the tropopause is reached. Thereafter, temperature variations cannot always be fitted by equation (3); or, where they can be fitted, the value of λ seems to be significantly smaller than that at lower altitudes.

2.4 Temperature variations at lower latitudes

As already remarked, temperature variations at lower latitudes cannot be fitted well by equation (3). An example is shown for 15 deg latitude (Fig. 8); lower altitudes exhibit a more pronounced bimodality or some undefined behavior, depending on the altitude and latitude. Where bimodality is pronounced, a simpler nonlinear version of the model could work; where the behavior is more complex, it is not clear that simple modifications of the model will suffice.

2.5 Largest gradients in temperature variations

For any specified latitude, one can write from equation (3) that

$$\frac{dT}{dt} = \frac{(\eta L - T)}{\lambda}, \tag{8}$$

and compute $\frac{dT}{dt}$ using the measured values of λ and η. The result is shown in Fig. 9 for 50 deg latitude. The largest changes occur sometime in April as well as October, consistent with the expectation that weather changes are most rapid in these months of the year.

3 Interpretation, discussion and conclusions

A linear viscoelastic model seems to describe some gross features of temperature variations at mid-latitudes. A prominent qualitative characteristic of the model is the "memory" it incorporates. The most significant quantitative information concerning this memory is the parameter λ. The best fit to the data yields the result that λ decreases with increasing latitude, as shown in

Fig. 5. (Note that all the data correspond to a pressure altitude of 1000 *mb*. It is not inconceivable that a somewhat different behavior could result if one chose, instead of a constant pressure altitude at all latitudes, a height at each altitude corresponding to the maximum amplitude in temperature variations. This, however, is work for the future.) Our initial response to Fig. 5 was to think that the result was counterintuitive, on the simple-minded ground that the response to equatorial heating must be faster at lower latitudes by virtue of their nearness to the equator. However, the complex nature of the physical processes which give rise to memory is not well understood. The present model averages temperature over all longitudes – and, at any given latitude, over many years – and is thus similar in spirit to general circulation models. However, it is difficult to associate any single time constant for the general circulation processes between equatorial and polar latitudes. The information from the present study, that indeed the energy transfer process can be described by a time constant whose variation with the latitude is given in Fig. 5, must therefore be considered completely new.

To understand the parameter λ, let us imagine the energy balance for a strip of radius R centered on the globe at latitude ℓ, of width $\Delta\ell$ and a certain penetration depth to be determined. The disc gets heated by the component $cos\ell$ of the light radiated normal to the surface, loses (or gains) energy by convective leat loss to the neighboring discs as well as to the atmosphere, and stores (or loses) energy due to the imbalance between these two effects. This energy storage, associated with the cyclic heat-up and cool-down processes, occurs in the first few meters of the ground; as is well-known to divers and plumbers alike, one can identify a penetration depth, so to speak, below which the seasonal variations of temparature are not felt. The details of the energy loss (or gain) are not well understood, and the standard practice in the heat transfer literature (for instance) is to represent this effect by means of a term $h(T - T^*)$, where one's ignorance is lumped into the heat transfer coefficient h; here, T^* is a reference temperature. The energy balance then takes the form

$$\frac{\rho c \Lambda}{h} \frac{d(T - T^*)}{dt} + (T - T^*) = excess\ (deficit)\ energy\ supply, \qquad (9)$$

where ρ and c are the density and specific heat, respectively, of the soil in the upper few meters of earth's crust and Λ is the characteristic penetration of heat. It is clear that this charactersitic depth should be proportional to the normal component $cos(\ell)$ of the Sun's radiation at latitude ℓ. If we assume that the daily excess energy supply is proportional to the length of the day measured from its 12-hour average, $< L^* >$, then equation (9) coincides with equation (1) if

$$\lambda = \frac{\rho\, c\, \Lambda}{h}, \qquad (10)$$

from where one can expect that

$$\frac{\lambda}{cos\ell} = constant. \qquad (11)$$

Table 2 shows that this ratio is indeed a constant of about 64 days. With $\lambda = 64\ cos\ell$ and $A = A(\ell)$ given by equation (2), equation (7) is an explicit formula for $\eta = \eta(\ell)$ whose graph resembles that shown in Fig. 5.

Daily variations in temperature are caused mainly by ground absorption and release of heat. We are describing the local heating of a large body by the periodic motion of a localized heat source. The thermal conductivity of the ground and its heat capacity are not so great, so (to a first approximation) it gives up at night what it received during the day. The imbalance at second order is responsible for the temperature lag; the ground gains heat as the length of the day increases and loses heat when it decreases. Inserting reasonable values for ρ, c and h (or, equivalently, a reasonable value for the so-called Biot number and thermal diffusivity of the soil), one gets a plausible estimate for the penetration depth to be of the order of a couple of meters. Clearly, more precise estimates of the penetration depth would depend on the latitude as well as on whether one is on land or ocean, but such considerations will be left for a future date.

An alternative form of equation (3) is

$$T(t,\ell) = A(\ell)\eta(\ell)cos\omega(t - \phi(\ell)) \tag{12}$$

where $\phi(\ell)$ is the phase lag at latitude ℓ. It is easily seen that $\phi = \omega^{-1}tan^{-1}(\omega\lambda)$, which means that the phase lag is determined entirely by λ. Taking $\lambda = 64cos\ell$ from equation (11) and Table 2, we may write

$$\phi(\ell) = \omega^{-1}tan^{-1}(64\omega cos\ell). \tag{13}$$

The behavior of ϕ with respect to latitude is shown in Fig. 10. A reasonably good empirical fit for mid-latitudes is

$$\phi(\ell) = 52 - 13tan\ell \tag{14}$$

where ϕ is expressed in days and ℓ in degrees.

Equation (12) gives an explicit expression for mean temperature variations as a function of latitude and time of the year. The amplitude of these variations is the product $A\eta$, where A can be obtained from the empirical fit (2), and η from Fig. 5; as already mentioned, the phase lag can be approximated by equation (13). For consistency, the product $A\eta$ must be fitted by equation (7), which it more or less does.

Two further comments are in order. First, equation (1) is tantamount to assuming that the heating of the earth by the sun can be represented by the term $\eta L(t)$. In principle, this term can be computed exactly from known information on the solar radiation arriving at the earth, earth's orientation with respect to the sun (i.e., the angle at which solar radiation hits different latitudes of the earth), and so forth. We have not attempted to do this. In effect, the "viscosity coefficient" η lumps these factors into a single number. Secondly, we have so far examined temperature variations at a latitude about its mean $< \theta >$ at the latitude; the model therefore has nothing to say about the latter. If one imagines that equation (1)–obviously without the derivative term–holds for the mean temperature $< \theta >$ as well, it is clear that one can define another constant η^* given by

$$\eta^* = \frac{< \theta >}{< L^* >}, \tag{15}$$

where the denominator $< L^* >$, as already remarked, is approximately 12 hours at all latitudes. There is no reason to expect that this new coefficient η^* will be related to η; in fact, a moment's consideration shows that η^* will be negative for upper latitudes. Empirically, we may observe from Fig. 11 that the variation of η^* as a function of $cos(\ell)$ is roughly linear with the fit given by

$$\eta^* = 4.53\ell - 2.33. \tag{16}$$

The ratio $\frac{\eta^*}{\eta}$ is around 0.5 for low latitudes, becoming zero at about 55 deg and negative further towards the pole.

The combination of equations (17), (16), (12) and (13) leads to an expression for the total temperature θ as a function of time and latitude.

Acknowledgements.

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Table 1. The coefficient λ for a few cities in the US. Despite of the scatter, the trend towards larger λ for lower latitudes is unmistakable. Latitude are rounded off to the nearest degree. At low latitudes such as those of Miami, temperature variations are not fitted well by equation (3), and hence the value of λ is quite approximate.

city	latitude, deg	λ, days
Fairbanks, AL	64	20
Anchorage, AL	60	27
Minneapolis, MN	45	30
Burlington,VA	44	27
Boston, MA	42	38
Chicago, IL	42	34
New Haven, CT	41	39
Boulder, CO	40	36
Columbus, OH	40	34
Little Rock, AR	35	47
Jacksonville, FL	30	42
Miami, FL	26	57

Table 2. The ratio $\lambda/cos(\ell)$ at various latitudes. The ratio is a constant with a mean value of 63.8 and a standard deviation of about 3.1.

latitude ℓ, deg	$\lambda/cos(\ell)$
25	62.9
30	64.7
35	63.5
40	61.4
50	59.1
60	66.0
65	68.6

Figure captions

Figure 1: The variation of the length of the day, $L(t)$, at 50 deg latitude, as a function of the number of days counted with respect to the summer solstice. The data are from ref. 2. The full line is a cosine term which is seen to fit the data quite well. The deviations from the fit can be easily accommodated by including a few higher harmonics, but the effort was deemed unnecessary for the accuracy attempted here.

Figure 2: The variation of the coefficient A as a function of the latitude. The full line is given by equation (2).

Figure 3: The variation of T at 50 deg latitude as a function of the number of days, counted with respect to the summer solstice, showing that it can be fitted quite well by equation (3). The observational data are from ref. 4, which were obtained by taking a five-year average between 1955 and 1960. It is not entirely clear that the five-year averaging is adequate for achieving stationarity, but the authors of ref. 4 remark that scanty evidence available over a period of some twenty years are substantially the same as their five-year averages. This may be correct because the averages are obtained over many weather stations located on any given latitude. Note, however, that the weather stations were not located on a uniform grid on the globe (see, for example, Figs. 1a and 1b of ref. 4), which may introduce some unknown, albeit small, bias in the average estimates. (An estimate of fluctuations about the averages will be provided later for local stations.) Similar data for the southern hemisphere are rather scanty.

Figure 4: A comparison between the amplitude of the sine/cosine fit to the temperature data and half the difference between the maximum and minimum temperatures in the real data. Their good agreement is a measure of the goodness of the sine/cosine fit to temperature variations in mid-latitudes.

Figure 5: The parameters λ and η as a function of the latitude. (Note the

scale change between the two parameters.) All data correspond to a pressure altitude of 1000 mb.

Figure 6: A comparison of the temperature variations at two substantially different latitudes. The data show without ambiguity that the warmest day occurs later in the year at lower latitudes than at higher latitudes.

Figure 7: The variation of $T(t)$ for the city of New Haven as a function of the number of days, counted with respect to the summer solstice, showing that it can be fitted well by equation (3). The observational data are from ref. 5 obtained over many (but unknown number of) years. A twelve year average shows that the standard deviation is of the order of 4^o C around the mean; a similar standard deviation on the date of occurrence of the coldest (or warmest) day of the year is of the order of ten days.

Figure 8: Temperature variations at 15 deg latitude, showing the limitations of the simple model represented by equation (1).

Figure 9: The temporal derivative of the temperature $T(t)$ through the year, as computed for 50 deg latitude using equation (8).

Figure 10: The variation of the phase lag ϕ with respect to latitude.

Figure 11: The variation of the coefficient η^* as a function of the cosine of the latitude.

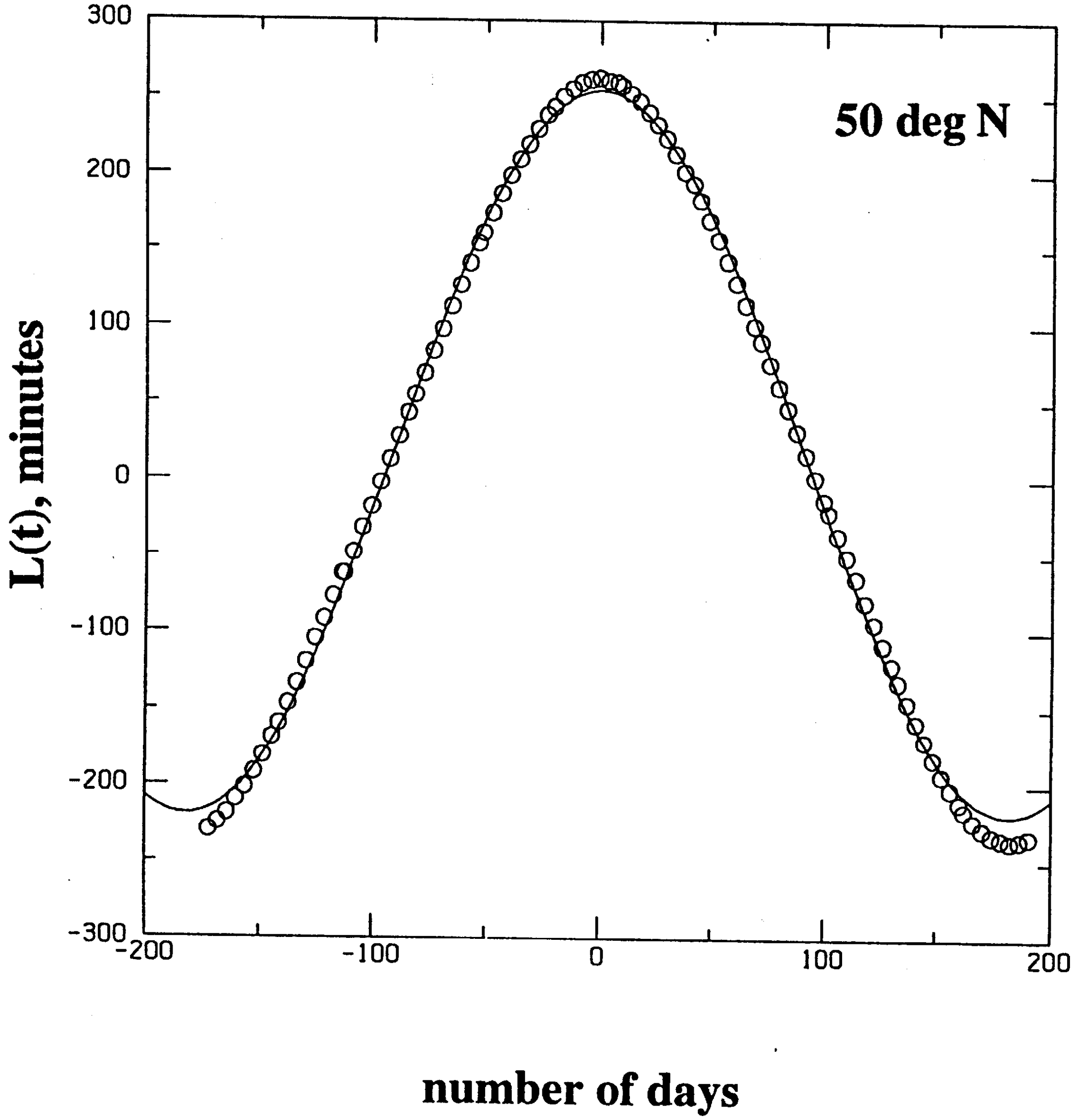

300
200
100
0
-100
-200
-300
-200
-100
0
100
200
50 deg N
L(t), minutes
number of days

Figure 1

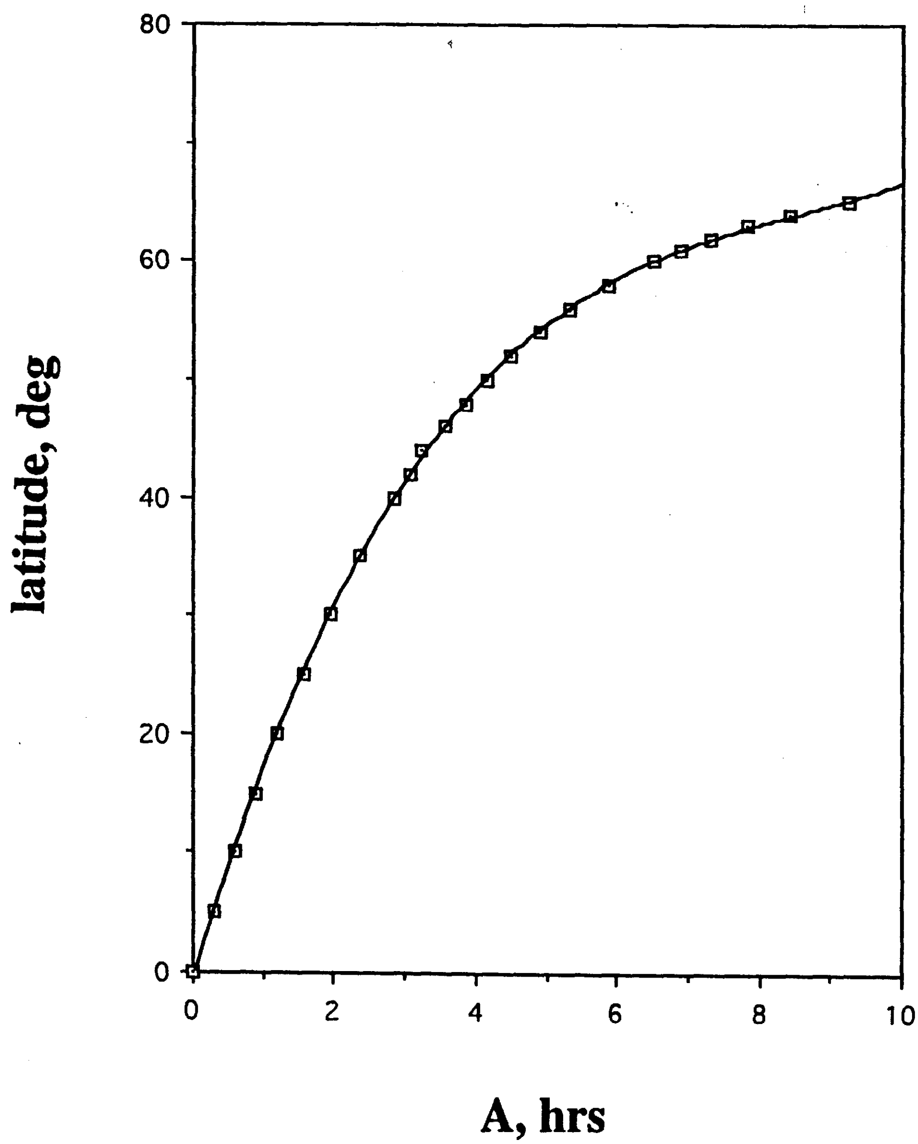


Figure 2

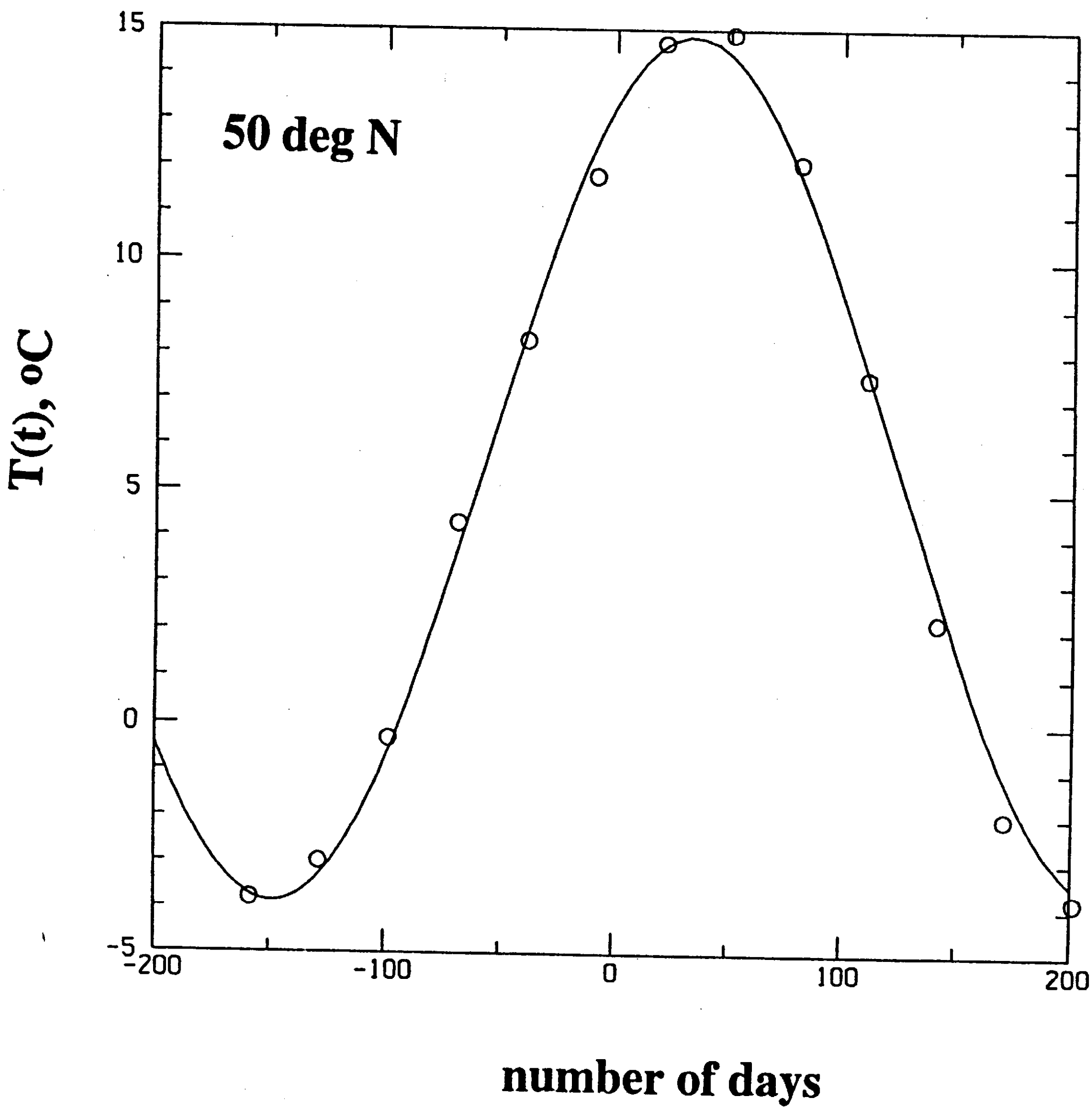


Figure 3

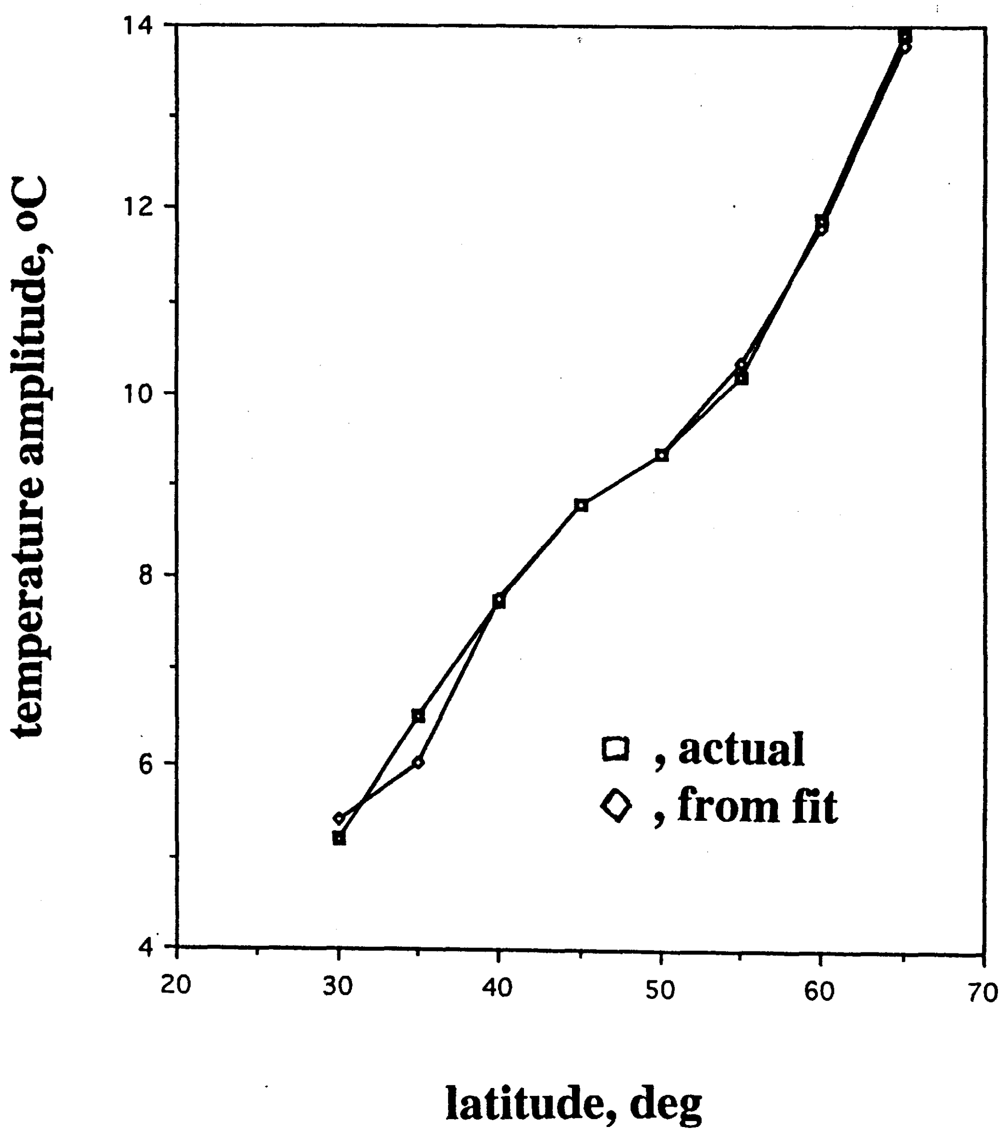

14
12
10
8
6
4
20
30
40
50
60
70
temperature amplitude, oC
latitude, deg
, actual
, from fit

fig 4

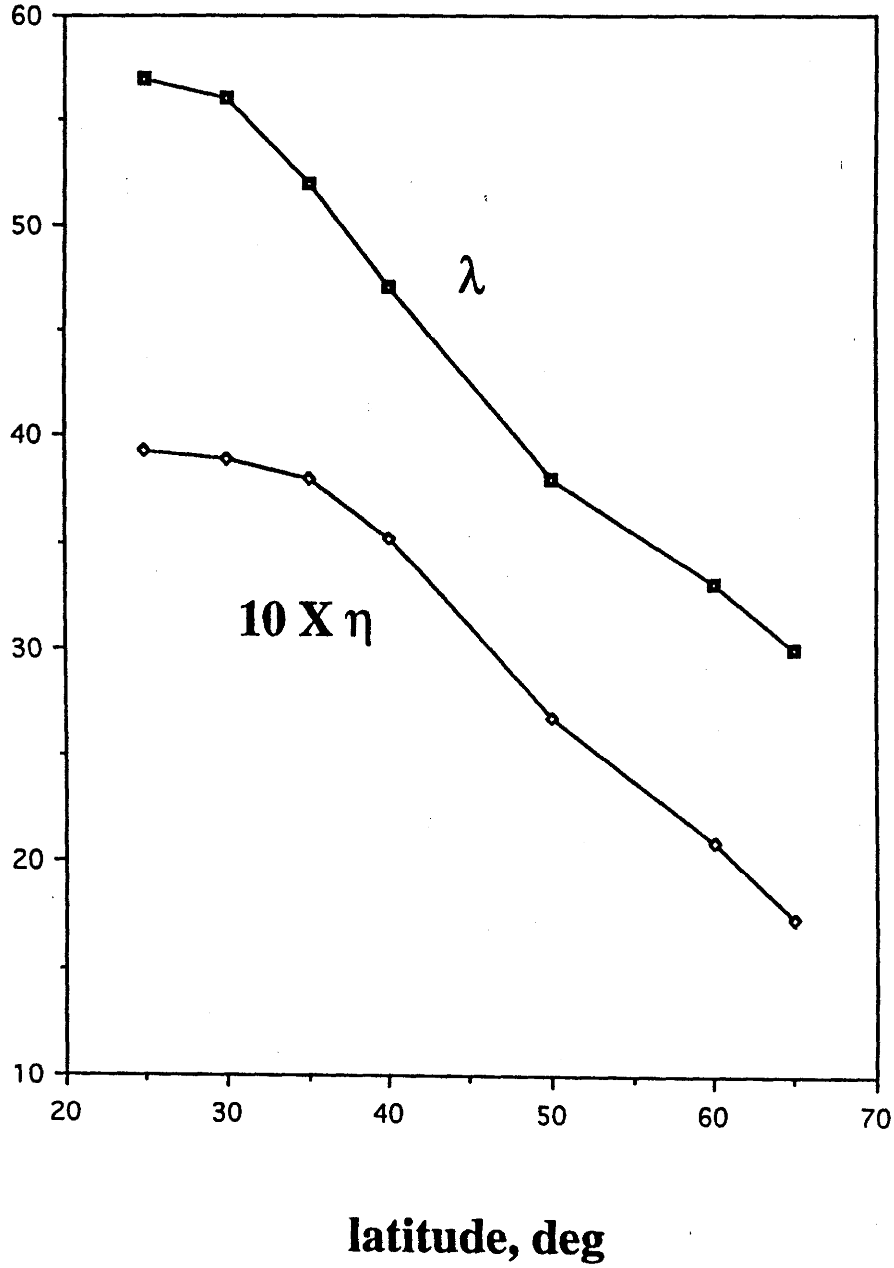
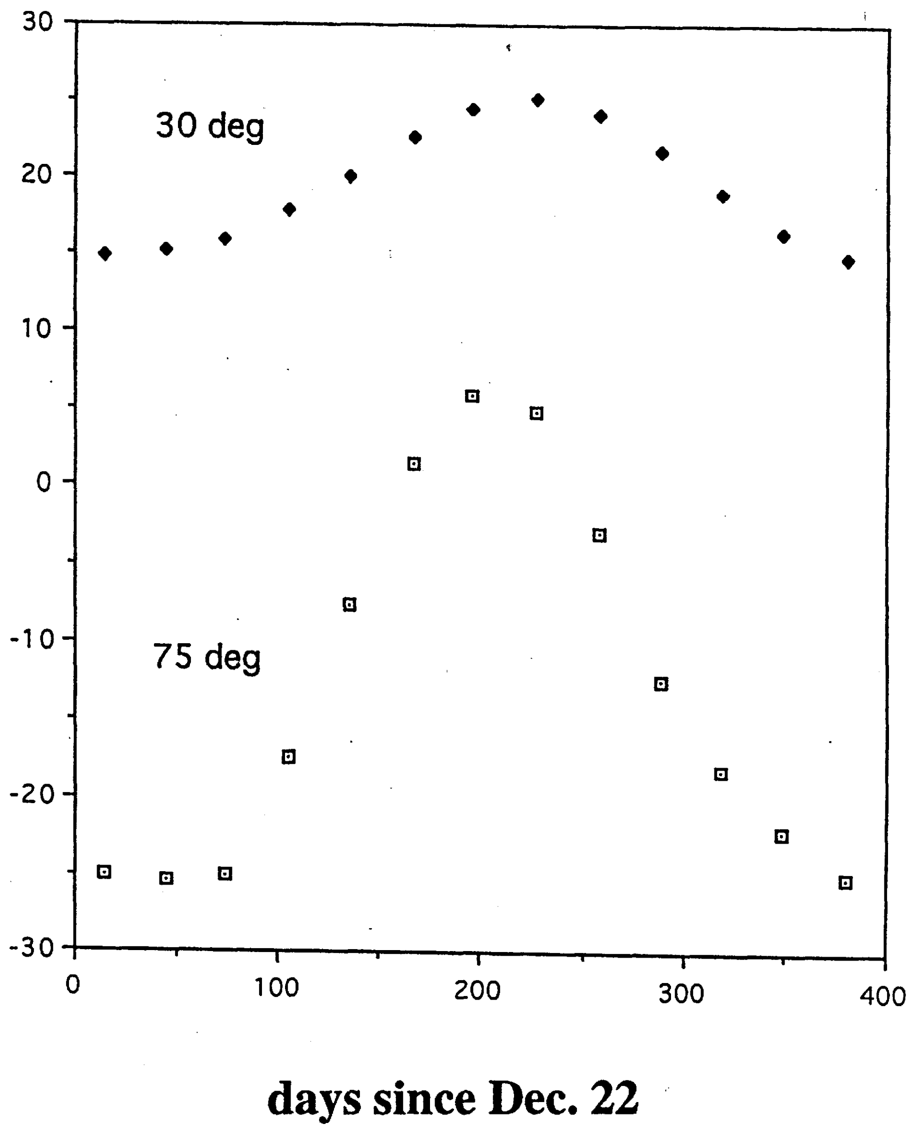


Figure 5


30
20
10
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-20
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0
100
200
300
400
30 deg
75 deg
days since Dec. 22

Figure 6

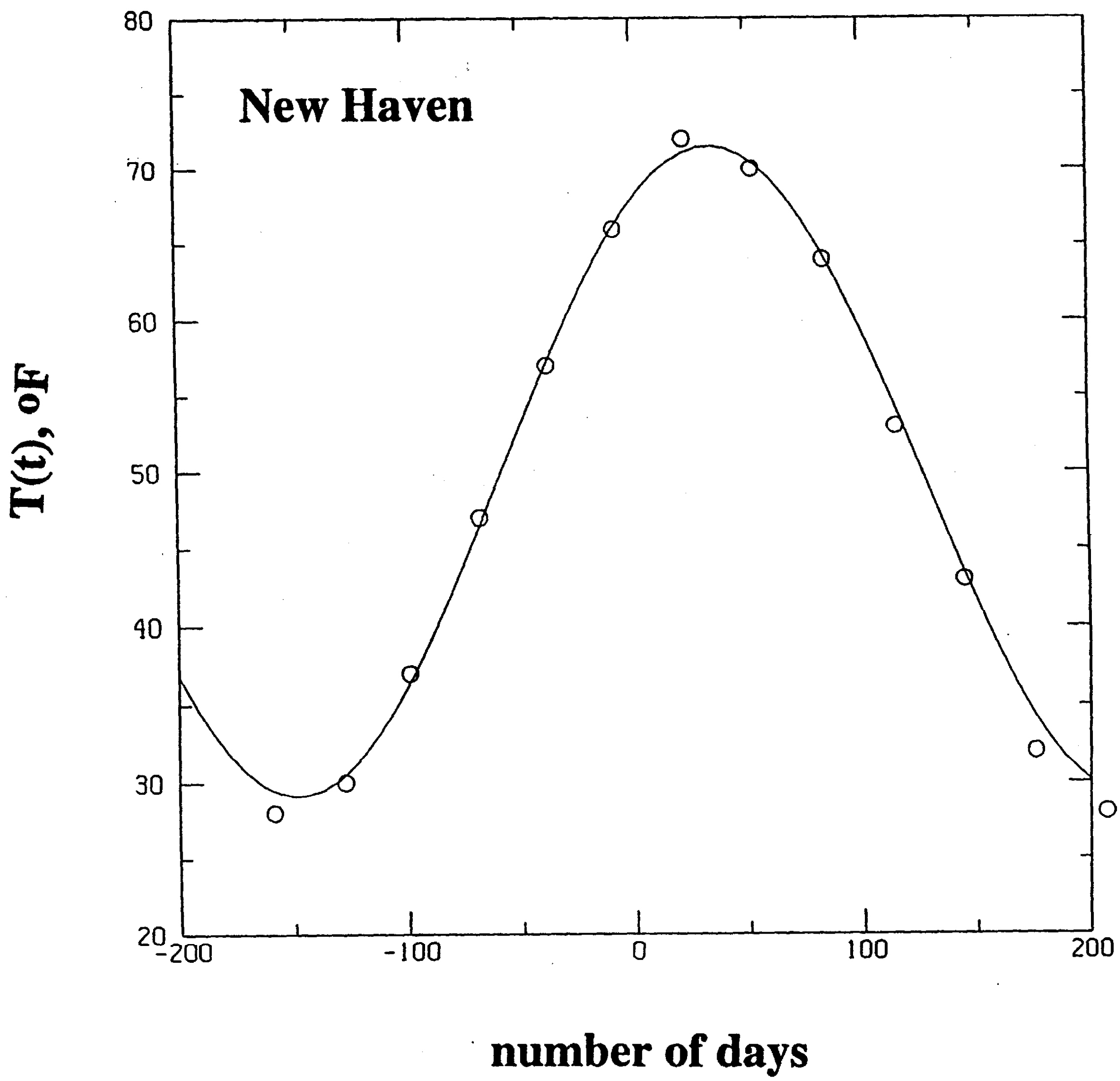
New Haven
80
70
60
50
40
30
20
-200
-100
0
100
200
T(t), oF
number of days

Figure 7

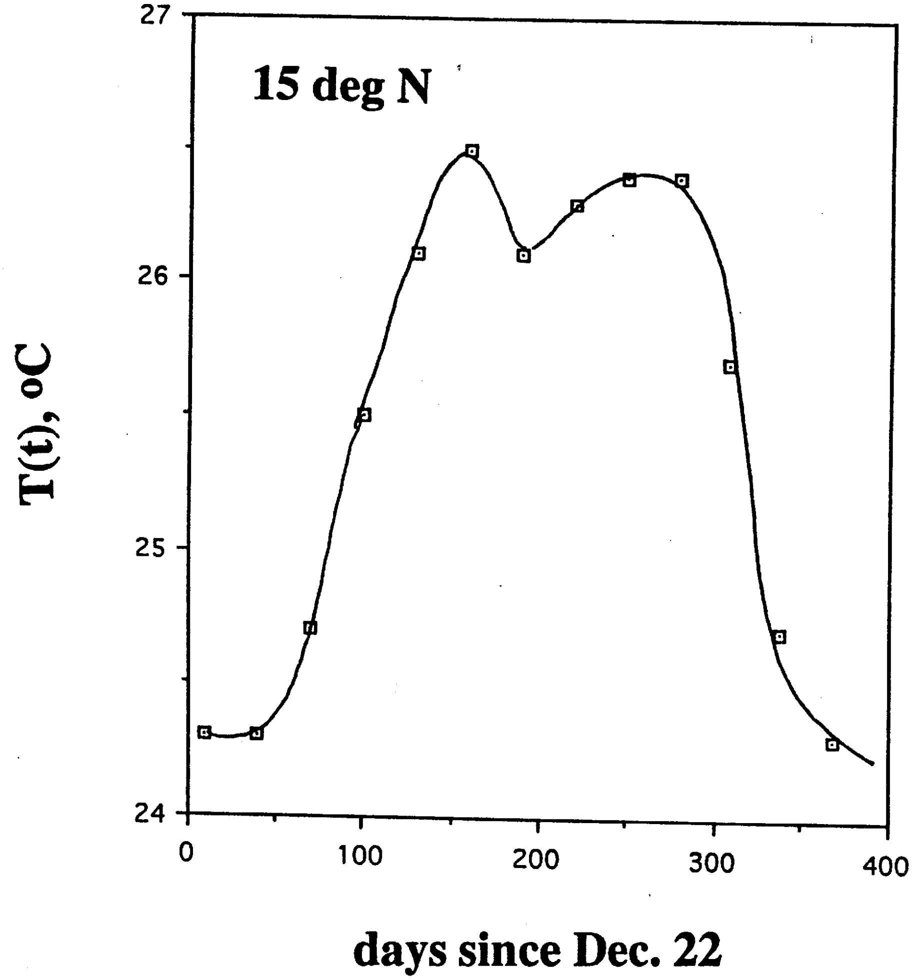

Figure 8

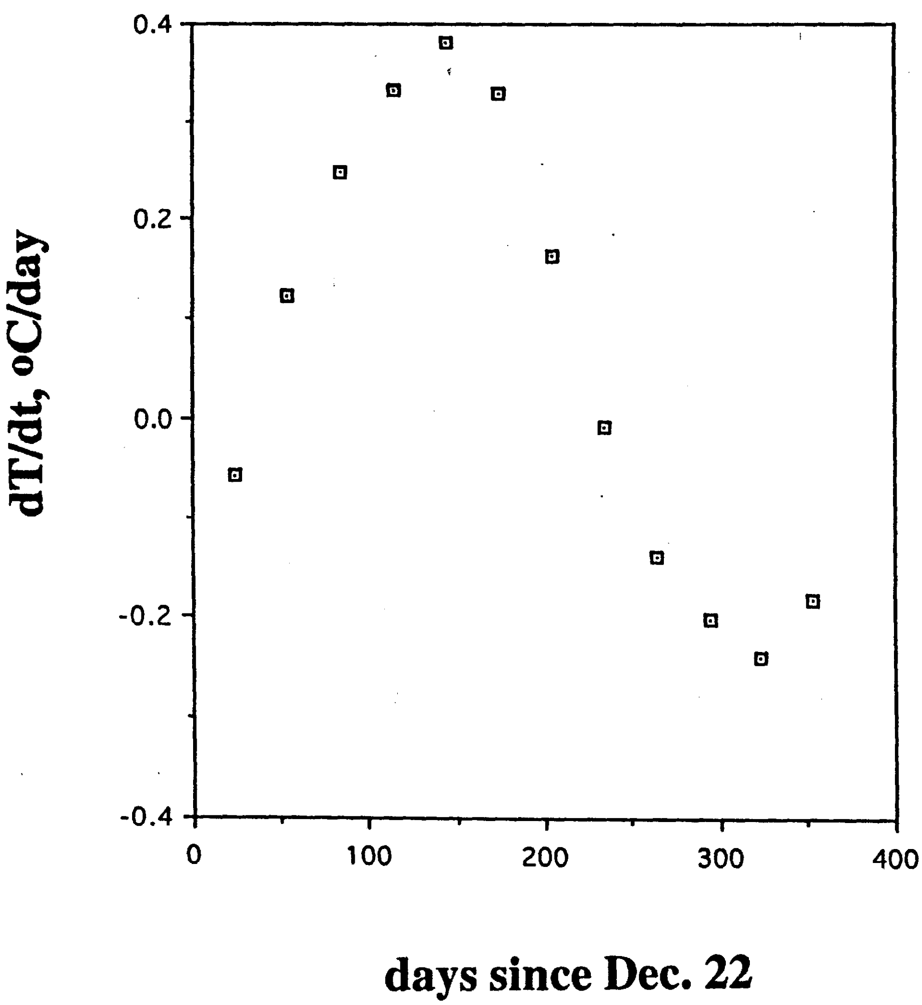

figure 9

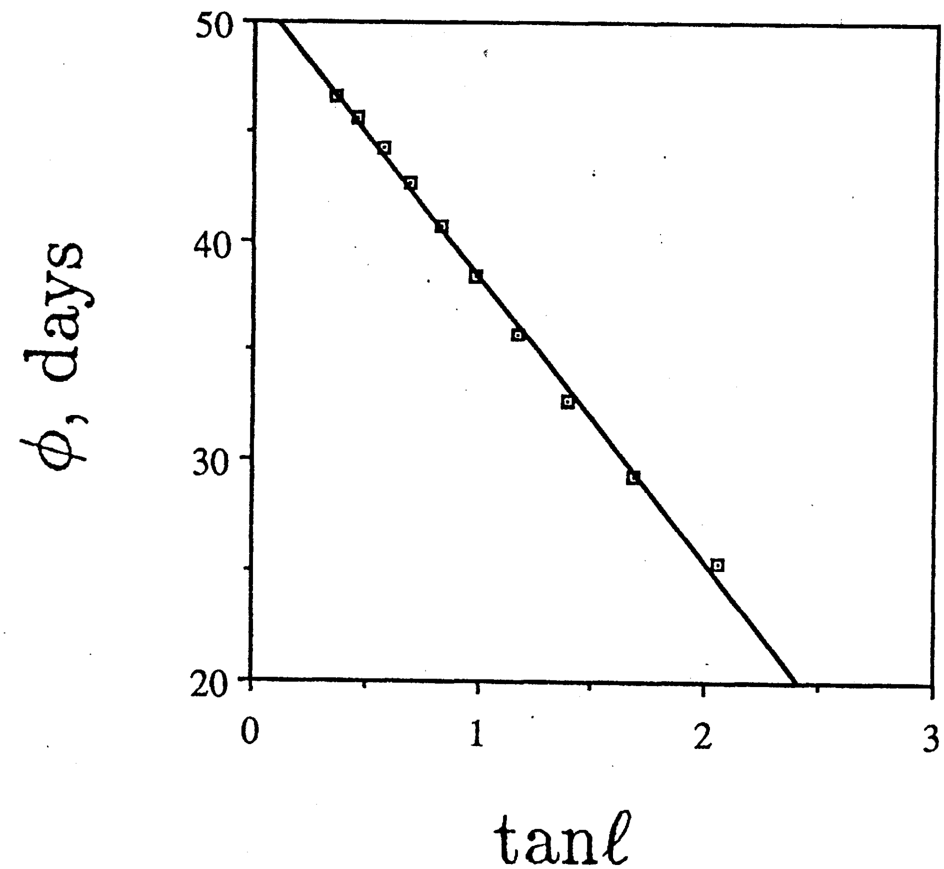

Figure 10

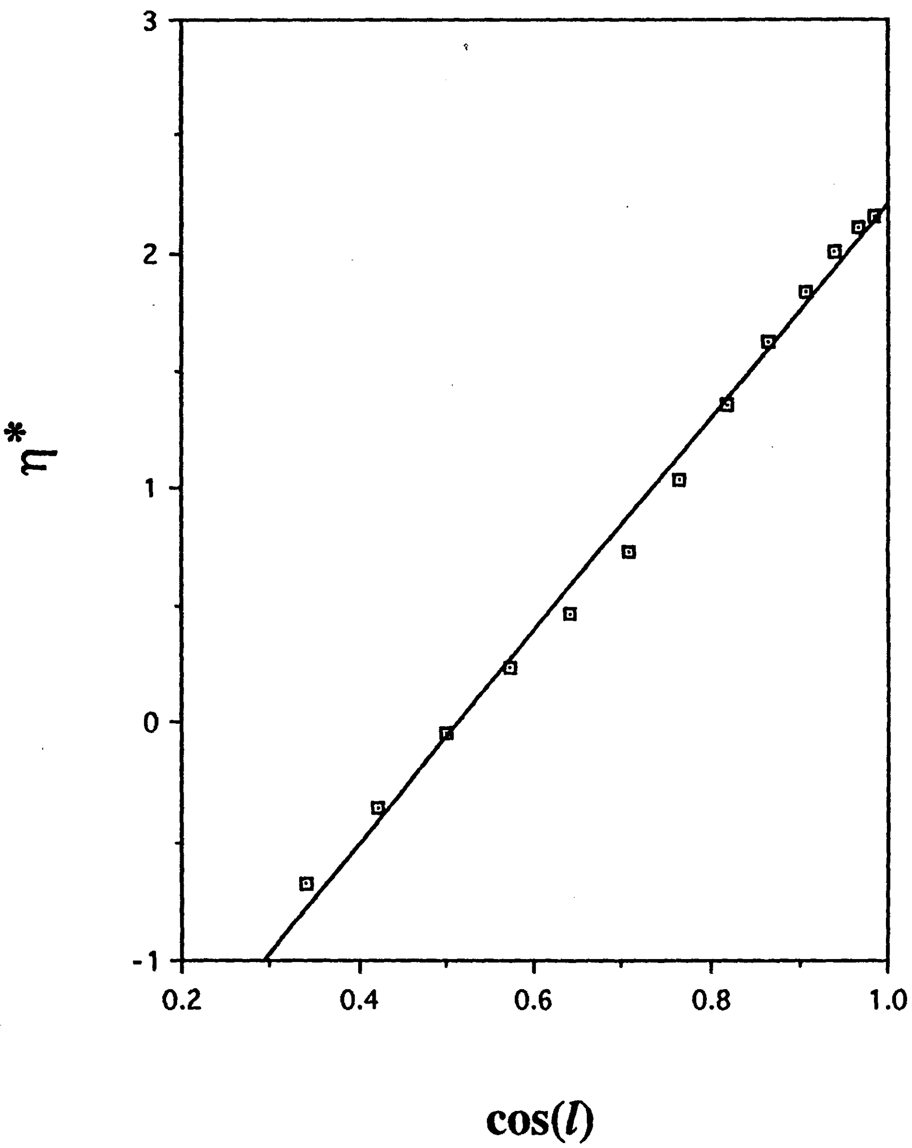

Figure 11

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Dept. 612-626-1558	Phone #
Fax # I WAIT FOR FURTHER INSTRUCTIONS	

Dan, at least the rejection was fast !! Sreeni

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Dear Dr Sreenivasan,

Thank you for submitting your manuscript, "A memory model for seasonal variations of temperature in mid-latitudes", which we are regretfully returning with this letter.

It is Nature's policy to return a substantial proportion of manuscripts without sending them to referees, so that they may be sent elsewhere without delay. Decisions of this kind are made by the editorial staff, often on the advice of regular advisers, when it appears that papers are unlikely to succeed in the competition for limited space.

Among the considerations that arise at this stage are the length of a manuscript, its likely interest to a general readership, the pressure on space in the various fields of Nature's interest and the likelihood that a manuscript would seem of great topical interest to those working in the same or related areas of science.

In this case, while fellow specialists will undoubtedly be interested in your results, I regret that we are unable to conclude that the paper provides the sort of important new insight into general issues, such as the physical processes determining the surface 'memory effect', that would excite the immediate interest of a wider audience of non-specialists. We therefore feel that the paper would find a more suitable outlet in a specialist journal, rather than Nature, and I am accordingly returning your manuscript.

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Assistant Editor

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