Mathematical Models and Methods in Applied Sciences
Vol. 12, No. 11 (2002) 1653–1690

ORIENTATION OF SYMMETRIC BODIES FALLING IN A SECOND-ORDER LIQUID AT NONZERO REYNOLDS NUMBER

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Received 10 May 2002
Communicated by G. P. Galdi

We study the steady translational fall of a homogeneous body of revolution around an axis a, with fore-and-aft symmetry, in a second-order liquid at nonzero Reynolds (Re) and Weissenberg (We) numbers. We show that, at first order in these parameters, only two orientations are allowed, namely, those with a either parallel or perpendicular to the direction of the gravity $\mathbf{g}$. In both cases the translational velocity is parallel to $\mathbf{g}$. The stability of the orientations can be described in terms of a critical value E_c for the elasticity number $E = \mathrm{We}/\mathrm{Re}$, where E_c depends only on the geometric properties of the body, such as size or shape, and on the quantity $(\Psi_1 + \Psi_2)/\Psi_1$, where Ψ_1 and Ψ_2 are the first and second normal stress coefficients. These results are then applied to the case when the body is a prolate spheroid. Our analysis shows, in particular, that there is no tilt-angle phenomenon at first order in Re and We.

Keywords: Sedimentation; orientation; second-order fluid; torque; tilt angle.

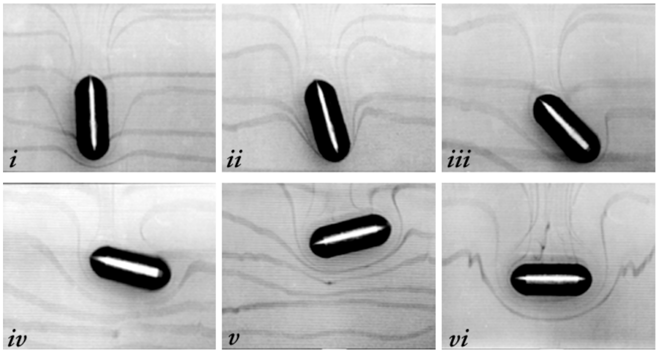
1. Introduction

It is a well-established experimental fact that homogeneous particles in the shape of bodies of revolution around an axis a (say) with fore-and-aft symmetry[a] (like cylinders, round ellipsoids, etc., of constant density), when dropped in a quiescent viscous liquid will eventually reach a steady state that is purely translatory (no spin), and with a forming an angle with respect to the gravity $\mathbf{g}$, that depends on the weight of the body, on its geometric properties (like being prolate or oblate in shape), and on the physical properties of the liquid (viscosity, inertia, non-Newtonian characteristics, etc.).[1,4,6,19,21,25,28] In particular, if the liquid is viscous and Newtonian, (homogeneous) cylinders or prolate spheroids will always reach an equilibrium orientation with a orthogonal to the gravity, no matter what their initial orientation[1,25]; see Fig. 1(a). It is important to observe that, in these experiments, the Reynolds number $\mathrm{Re} = Vd/\nu$ can be very small. This is due to the fact that, typically, the product of the terminal speed V of the body and its characteristic length d is small compared to the kinematical viscosity ν of the liquid. For example, in a 85% aqueous solution of glycerine for a body having $d = 0.5$ cm and $V = 1$ cm/sec, the corresponding Reynolds number is $\mathrm{Re} = 0.44$. However, despite the smallness of the Reynolds number involved, these phenomena are genuinely *nonlinear*, and originate from the *inertia* of the liquid. In fact, if we make the liquid very viscous (99.9% of glycerine, say), so that the Reynolds number reduces approximately to zero and inertial effects can be neglected, then it is observed that the body will always keep its initial orientation with $\mathbf{g}$.[28] In other words, *all* orientations are admissible at $\mathrm{Re} = 0$.[b]

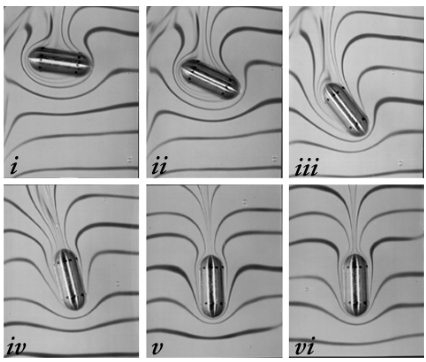
If a small amount of polymer is added to the Newtonian liquid (typically, a 0.5%–2% aqueous solution), the situation changes dramatically and the final orientation may be completely different from that observed for a Newtonian liquid at nonzero Re. Detailed experimental studies were performed on slender cylinders sedimenting in aqueous solution polyacrylamide of different concentration; see Refs. 4 and 19. In these experiments Re is much smaller than the corresponding dimensionless elasticity parameter, so that the effect due to the inertia of the liquid can be neglected, a fact that typically happens when particles are very light, as in a fiber suspension.

The final orientation of all particles is observed to be with their broadside *parallel* to $\mathbf{g}$; see Fig. 1(b). This is quite remarkable, since it is in sharp contrast with the Newtonian case where, as we described before, a long particle will reach an equilibrium configuration with its broadside *perpendicular* to $\mathbf{g}$; see Fig. 1(a).

[a]By this latter we mean that there is a plane Π orthogonal to a that is of symmetry for $\mathcal{B}$.
[b]The conservation of the initial orientation depends, of course, on the elapsed observation time. This means that, if we wait a sufficiently long time (depending on the viscosity), inertia will eventually prevail and the body will turn with a perpendicular to $\mathbf{g}$. In practice, we would need a sufficiently tall liquid container in which to drop the body, in order to observe a significant deviation from its initial orientation.



(a)



(b)

Fig. 1. Orientation of cylinders with round ends in a Newtonian liquid at nonzero Reynolds number (a), and in a purely viscoelastic liquid at vanishingly small Reynolds number (b).[17] In the Newtonian case (a) the initial configuration is with the major axis parallel to the gravity (unstable) and the terminal configuration is with the major axis perpendicular to the gravity (stable). In the purely viscoelastic case (b) the situation is reversed.

A recent experimental study on the orientation of long particles sedimenting in viscoelastic liquid,[21] shows another remarkable feature. Let us call *tilt angle* the angle formed by the long axis of symmetry a of the particle with the horizontal, when the body reaches its final equilibrium orientation. In Ref. 21 it is found that for squared-off cylinders the tilt angle may vary continuously from $0°$ to $90°$, depending on the physical properties of the cylinder and on the concentration of polymeric liquid. The tilt angle is very stable and it is reached no matter how and where the cylinder is released. Analogous conclusions on the tilt angle were reached in the experiments on sedimentation of very light cylinders reported in Ref. 6.

Dimensionless numbers involved in all the above experiments may be very small. For example, for cylinders made of plastic, Teflon, aluminum and titanium, with length ~ 2 cm and diameter in the range 0.25–1 cm, it is found that Re varies from

0.016 to ~ 5, while the Weissenberg number We $= \lambda V/d$, with λ the *relaxation time*, ranges between 0.048 and ~ 0.3,[21] p. 580. Furthermore, "wall effects" play no role on the preferred orientation of the particle. (For example, in Ref. 6, the ratio of the length of the cylinders to the diameter of the container is of the order of 10^{-2}.)

The following qualitative explanation of the tilt angle phenomenon was more recently proposed in Ref. 18, and it is based on the competition between inertia and normal stress effects (see Fig. 2). In fact, if the liquid is purely Newtonian, (We $= 0$, say) inertial effects are relevant near the two stagnation points S_1 and S_2, where the pressure has a maximum. This generates a torque that tends to rotate the body with its broadside *perpendicular* to **g**, a fact first discovered by Lord Kelvin.[29] However, if the liquid is purely viscoelastic and the motion of the body is slow (Re $= 0$), the fluid rheology can be described by a second-order model, and in Ref. 18 it is then proved that the normal stresses on the body are compressive and are large at points of high shear A and B. Therefore, a viscoelastic torque is generated that tends to align the body with its broadside *parallel* to **g**. When *both* inertia and normal stress effects are present, the two torques will compete, and the tilt angle would be the equilibrium configuration arising from this competition. Since this can occur even when the two effects are very weak, it is suggested that a perturbation analysis could explain the result of the experiments. In Ref. 9 a numerical computation of the torque on a prolate spheroid translating in a second-order liquid, based on perturbation theory was carried out, with the objective of supporting the validity of this conjecture. However, this paper contains an error that invalidates the main results. Successively, in Ref. 15 results were reported of a direct 2D numerical simulation of the sedimentation of elliptic particles of eccentricity $e = 0.745$ in Oldroyd-B liquids at small Reynolds and Deborah (De) numbers ($\lesssim 1$). These results do not show the occurrence of the tilt angle. Rather, they suggest the existence of a critical value E_c for the elasticity number $E = \text{De}/\text{Re}$

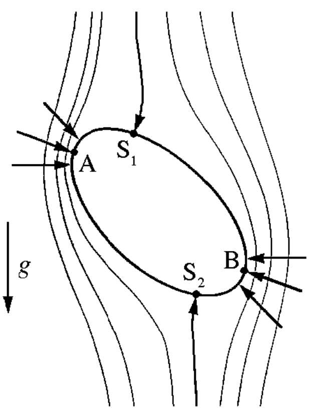


Fig. 2. Slow flow around an elliptic particle. At the two stagnation points S_1, S_2 the pressure is a maximum. Strong shears produce large normal stresses at points A and B.

with the following property: for $E < E_c$ the liquid behaves as Newtonian, in the sense that the final orientation of the ellipse is broadside-on for all values of the fall velocity V, while if $E > E_c$ the ellipse finally settles with its long axis vertical, provided V is not too large. In Ref. 15 it was also found that, if shear-thinning was incorporated in the Oldroyd-B model by using the Carreau–Bird law, the ellipse would eventually reach an orientation with a well-defined tilt angle between $0°$ and $90°$. This suggests that shear-thinning, more than normal stresses, could be the important parameter for the tilt angle phenomenon. This would also agree with the outcome of the experiments made in Ref. 6.

In view of these yet non-conclusive results, it seems appropriate to study the problem of orientation of symmetric particles by a rigorous mathematical analysis. Only partial quantitative results along these lines are available in the literature, and only when either $\mathrm{We} = 0$ (purely Newtonian case[8,13,14]) or when $\mathrm{Re} = 0$ (purely viscoelastic case[3,5,11,19,20]). However, for the problem we are interested here, it is crucial to consider *both* Re and We to be *nonzero*.

In this paper we investigate the orientation of a homogeneous body $\mathcal{B}$ of revolution around a, possessing the fore-and-aft-symmetry, and slowly settling through an otherwise quiescent viscoelastic liquid, modeled as a second-order fluid. Specifically, we present a mathematical analysis aimed at finding all possible orientations of $\mathcal{B}$ at small and nonzero Reynolds and Weissenberg numbers. Our analysis relies upon the evaluation of the torque $\boldsymbol{\mathcal{M}}$ exerted by the liquid on the body. Specifically, taking a coinciding with the x_1-axis of a frame attached to $\mathcal{B}$, and, without loss, the translational velocity $\mathbf{U} = (U_1, U_2, 0)$, we show that (Sec. 2)

$$\boldsymbol{\mathcal{M}} = (\mathrm{Re}\,\mathcal{G}_{\mathcal{I}} + \mathrm{We}\,\mathcal{G}_{\mathcal{V}}^{(\varepsilon)})U_1U_2\mathbf{e}_3 + \boldsymbol{\mathcal{N}}(\mathrm{Re},\mathrm{We})\,. \tag{I}$$

Here $\mathbf{e}_3$ is a unit vector in the x_3-direction. $\mathcal{G}_{\mathcal{I}}$ and $\mathcal{G}_{\mathcal{V}}^{(\varepsilon)}$ are scalar quantities — that we call *inertial* and *viscoelastic torque coefficients*, respectively — depending on the geometric properties of $\mathcal{B}$, such as size or shape, but otherwise independent of its orientation. $\mathcal{G}_{\mathcal{V}}^{(\varepsilon)}$ depends also on the parameter $\varepsilon = 2(\Psi_1 + \Psi_2)/\Psi_1$, where Ψ_1 and Ψ_2 are the first and second normal stress coefficient. A typical variation range of ε is between 1.6 and 2.[16] Finally $\boldsymbol{\mathcal{N}}(\mathrm{Re},\mathrm{We})$ is a "remnant" that can be estimated as follows:

$$|\boldsymbol{\mathcal{N}}(\mathrm{Re},\mathrm{We})| \le C(\mathrm{Re}^{2-\eta} + \mathrm{We}^2)\,,$$

where η is arbitrary in $(0, 2)$ and C is a constant depending only on $\mathcal{B}$, ε and η, with $C \to \infty$ as $\eta \to 0$. The expression (I) for the torque is obtained as a special case of a general method for the evaluation of the torque that is presented in Sec. 1, and which applies, in principle, to different or much more complicated liquids. Taking into account that in an equilibrium orientation of $\mathcal{B}$, the torque, must be zero, from (I) we draw a number of consequences. Actually, we prove (Sec. 3) that, provided $\mathcal{G} \equiv \mathrm{Re}\,\mathcal{G}_{\mathcal{I}} + \mathrm{We}\,\mathcal{G}_{\mathcal{V}}^{(\varepsilon)}$ is not zero, at first order in Re and We there are *only two possible orientations*, namely those for which $\mathbf{g}$ is either parallel or orthogonal to a. If, however, $\mathcal{G} = 0$, then *all* orientations are possible at first order. This excludes the possibility of a tilt angle. Moreover, if $\mathcal{G} \neq 0$, the stability to small disorientations of the two possible orientations is related to the sign of $\mathcal{G}$. We see that if $\mathcal{G} > 0$, then the orientation with **g** perpendicular to a is stable and the other is unstable. The reverse holds if $\mathcal{G} < 0$. These results are then analyzed in more details in Sec. 3, in the case when $\mathcal{B}$ is a prolate spheroid of eccentricity $e \in [0,1]$. In this case it turns out that $\mathcal{G}_{\mathcal{I}}$ is a computable *negative* function of $e \in (0,1)$ that vanishes at $e = 0,1$. Moreover, $\mathcal{G}_{\mathcal{V}}^{(\varepsilon)}$ is a *positive* function of $e \in (0,1)$ vanishing at $e = 0,1$, *provided* $\varepsilon \gtrsim 1$. So, in this range of ε, the two torques (inertial and non-Newtonian) are in competition, as expected. Another interesting feature is that for $\varepsilon = 1.8$ (a commonly accepted value for polymer solutions[16] p. 516), $\mathcal{G}_{\mathcal{V}}^{(\varepsilon)}$ is almost five times bigger than $|\mathcal{G}_{\mathcal{I}}|$, for $e \sim 0.85$. Concerning the stability of orientation, we show that it can be formulated in terms of a critical elasticity number defined as $E_{\rm c} = E_{\rm c}(e,\varepsilon) \equiv \frac{|\mathcal{G}_{\mathcal{I}}|}{\mathcal{G}_{\mathcal{V}}^{(\varepsilon)}}$. Specifically, if $E < E_{\rm c}$ the liquid behaves as Newtonian, that is the orientation with a perpendicular to **g** is stable with respect to "quasi-steady" small disorientations, while if $E > E_{\rm c}$, the other is stable. In the last section, we compare our results with the experiments performed in Ref. 21 with cylinders with round ends. They are in a good agreement for $\varepsilon = 1.8$.

The paper ends with the appendices that include all the technical proofs needed to obtain our main results.

2. Calculation of the Torque

The physical mechanism responsible for the orientation of a rigid body moving in a liquid by translational motion is the torque $\boldsymbol{\mathcal{M}}$ exerted by the liquid on the body. The objective of this section is to furnish a general method to evaluate $\boldsymbol{\mathcal{M}}$, at first order, for the case of a body of revolution with fore-and-aft symmetry.

Assume that a body $\mathcal{B}$ is moving in a viscous liquid $\mathcal{L}$, with a constant translational velocity **V**. The appropriate equations of motion can be written in a nondimensional form as follows

$$\left.\begin{array}{l}\mathrm{Re}\,\mathbf{v}\cdot\mathrm{grad}\,\mathbf{v} = \mathrm{div}\,\mathbf{T}_N(\mathbf{v},p) + \lambda\,\mathrm{div}\,\mathbf{S}(\mathbf{v}) \\ \mathrm{div}\,\mathbf{v} = 0\end{array}\right\}\ \text{in } \Omega$$
$$\mathbf{v} = 0 \quad \text{at } \Sigma \equiv \partial\Omega\,, \tag{2.1}$$
$$\lim_{|x|\to\infty} \mathbf{v}(x) = -\mathbf{U}\,.$$

Here **v** and p are (nondimensional) velocity and pressure fields of $\mathcal{L}$, Ω is the infinite region occupied by $\mathcal{L}$, i.e. the complement of $\mathcal{B}$, $\mathbf{U} = \mathbf{V}/V_0$, and

$$\mathrm{Re} = \frac{\rho V_0 d}{\mu} \quad \text{(Reynolds number)}$$

with V_0 scaling velocity, d diameter of $\mathcal{B}$, ρ the density of $\mathcal{L}$, and μ the shear viscosity coefficient. Moreover, $\mathbf{T}_N$ denotes the Newtonian stress tensor, namely,

$$\mathbf{T}_N = -p\mathbf{I} + 2\mathbf{D}(\mathbf{v})\,,$$

with

$$\mathbf{D}(\mathbf{v}) = \frac{1}{2}(\mathrm{grad}\,\mathbf{v} + (\mathrm{grad}\,\mathbf{v})^{\mathrm{T}})\,.$$

Finally, λ is a (nondimensional) parameter related to the non-Newtonian character of $\mathcal{L}$ and $\mathbf{S}$ is the non-Newtonian part of the stress tensor. Thus, the *total* stress is given by

$$\mathbf{T} = \mathbf{T}_N + \lambda\mathbf{S}\,.$$

The total torque $\boldsymbol{\mathcal{M}}$ exerted by $\mathcal{L}$ on $\mathcal{B}$ is given by

$$\boldsymbol{\mathcal{M}} \equiv -\int_\Sigma \mathbf{x} \times \mathbf{T} \cdot \mathbf{n}\,, \tag{2.2}$$

where $\mathbf{n}$ is the unit normal at Σ, directed toward $\mathcal{B}$. Our objective is to compute $\boldsymbol{\mathcal{M}}$ at first order in Re and λ. To this end, we introduce the fields $\mathbf{H}^{(i)}$, $P^{(i)}$, $i = 1, 2, 3$, defined as follows[14]:

$$\left.\begin{aligned} &\mathrm{div}\,\mathbf{T}_N(\mathbf{H}^{(i)}, P^{(i)} = \mathbf{0} \\ &\mathrm{div}\,\mathbf{H}^{(i)} = 0 \end{aligned}\right\} \text{in } \Omega \\ \mathbf{H}^{(i)} = \mathbf{e}_i \times \mathbf{x} \quad \text{at } \Sigma \\ \lim_{|x|\to\infty} \mathbf{H}^{(i)} = \mathbf{0} \tag{2.3}$$

where $\{\mathbf{e}_i\}$ is an orthonormal basis in $\mathbb{R}^3$. Multiplying $(2.3)_1$ by $\mathbf{H}^{(i)}$, integrating by parts over Ω and using $(2.3)_{2,3,4}$ we find

$$\mathcal{M}_i = 2\int_\Omega \mathbf{D}(\mathbf{v}) : \mathbf{D}(\mathbf{H}^{(i)}) + \lambda \int_\Omega \mathbf{S} : \mathbf{D}(\mathbf{H}^{(i)}) + \mathrm{Re}\int_\Omega \mathbf{v} \cdot \mathrm{grad}\,\mathbf{v} \cdot \mathbf{H}^{(i)}\,. \tag{2.4}$$

The first integral on the right-hand side of this relation can be evaluated by multiplying $(2.3)_1$ by $\mathbf{v} + \mathbf{U}$ and integrating by parts over Ω (see Ref. 20). We get

$$2\int_\Omega \mathbf{D}(\mathbf{v}) : \mathbf{D}(\mathbf{H}^{(i)}) = \mathbf{U} \cdot \int_\Sigma \mathbf{T}(\mathbf{H}^{(i)}) \cdot \mathbf{n}\,. \tag{2.5}$$

From (2.2), (3.3) and (2.5) we thus obtain

$$\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{M}}^S + \mathrm{Re}\,\boldsymbol{\mathcal{M}}^I + \lambda\boldsymbol{\mathcal{M}}^{NN}\,, \tag{2.6}$$

where, for $i = 1, 2, 3$,

$$\begin{aligned} \mathcal{M}_i^S &= -\mathbf{U} \cdot \int_\Sigma \mathbf{T}(\mathbf{H}^{(i)}) \cdot \mathbf{n}\,, \\ \mathcal{M}_i^I &= -\int_\Omega \mathbf{v} \cdot \mathrm{grad}\,\mathbf{v} \cdot \mathbf{H}^{(i)}\,, \\ \mathcal{M}_i^{NN} &= -\int_\Omega \mathbf{S}(\mathbf{v}) : \mathbf{D}(\mathbf{H}^{(i)}) \end{aligned} \tag{2.7}$$

are the torque in the Stokes approximation (i.e. Re $= \lambda = 0$), the torque due to inertia, and the torque due to the non-Newtonian character of $\mathcal{L}$, respectively.

We now denote by $(\mathbf{v}_S, p_S)$ and by $(\mathbf{v}_{NS}, p_{NS})$ the solutions to (2.1) with Re $=\lambda=0$ and $\lambda=0$, respectively. We also set

$$\mathbf{u}=\mathbf{v}-\mathbf{v}_{NS}\,,\quad \mathbf{w}=\mathbf{v}_{NS}-\mathbf{v}_S$$

and

$$\begin{aligned}\mathcal{M}_i^{0,I}&=-\int_\Omega \mathbf{v}_S\cdot\operatorname{grad}\mathbf{v}_S\cdot\mathbf{H}^{(i)}\,,\\ \mathcal{M}_i^{0,NN}&=-\int_\Omega \mathbf{S}(\mathbf{v}_S):\mathbf{D}(\mathbf{H}^{(i)})\,.\end{aligned}\tag{2.8}$$

From (2.6) we thus get

$$\boldsymbol{\mathcal{M}}=\boldsymbol{\mathcal{M}}^{0,I}+\mathrm{Re}\,\boldsymbol{\mathcal{M}}^{0,I}+\lambda\boldsymbol{\mathcal{M}}^{0,NN}+\boldsymbol{\mathcal{N}}\,,\tag{2.9}$$

where

$$\begin{aligned}\boldsymbol{\mathcal{N}}&=\mathrm{Re}(\boldsymbol{\mathcal{M}}^{I}-\boldsymbol{\mathcal{M}}^{0,I})+\lambda(\boldsymbol{\mathcal{M}}^{0,NN}-\boldsymbol{\mathcal{M}}^{0,NN})\\ &\equiv \mathrm{Re}\,\boldsymbol{\mathcal{N}}_1+\lambda\boldsymbol{\mathcal{N}}_2\,.\end{aligned}$$

By a straightforward calculation we find

$$\begin{aligned}\mathcal{N}_{1i}&=-\int_\Omega[(\mathbf{u}+\mathbf{w})\cdot\operatorname{grad}\mathbf{v}+\mathbf{v}_S\cdot\operatorname{grad}(\mathbf{u}+\mathbf{w})]\cdot\mathbf{H}^{(i)}\,,\\ \mathcal{N}_{2i}&=-\int_\Omega[\mathbf{S}(\mathbf{v})-\mathbf{S}(\mathbf{v}_{NS})]:\mathbf{D}(\mathbf{H}^{(i)})-\int_\Omega[\mathbf{S}(\mathbf{v}_{NS})-\mathbf{S}(\mathbf{v}_S)]:\mathbf{D}(\mathbf{H}^{(i)})\,.\end{aligned}\tag{2.10}$$

From (2.10) it is *expected* that both $\boldsymbol{\mathcal{N}}_1$ and $\boldsymbol{\mathcal{N}}_2$ should vanish as Re, $\lambda\to 0$, i.e.

$$\boldsymbol{\mathcal{N}}=o(\mathrm{Re})+o(\lambda)\quad \text{as Re},\lambda\to 0\,.\tag{2.11}$$

In this case, from (2.9) we deduce that, at first order in Re, λ

$$\boldsymbol{\mathcal{M}}=\boldsymbol{\mathcal{M}}^{S}+\mathrm{Re}\,\mathcal{M}^{0,I}+\lambda\boldsymbol{\mathcal{M}}^{0,NN}\,.\tag{2.12}$$

The above considerations apply to any body $\mathcal{B}$ (and to any liquid $\mathcal{L}$). Now we would like to consider the special case when $\mathcal{B}$ is a homogeneous[c] body of revolution around an axis a (say) with fore-and-aft symmetry. By this latter we mean that there is a plane orthogonal to a, that is of symmetry for $\mathcal{B}$. In such a case, it is well known that (see Sec. 5-5 of Ref. 14)

$$\int_\Sigma \mathbf{T}(\mathbf{H}^{(i)})\cdot\mathbf{n}=\mathbf{0}\,,\quad i=1,2,3\,.$$

This fact has two main consequences. The first (obvious) is:

$$\boldsymbol{\mathcal{M}}^{S}=\mathbf{0}\,,\tag{2.13}$$

and the second is (see Chap. V of Ref. 10):

$$\|\operatorname{grad}\mathbf{H}^{(i)}\|_s<\infty\,,\quad \text{for all } s\in(1,\infty)\,,\quad i=1,2,3\,.\tag{2.14}$$

[c]That is, the density of $\mathcal{B}$ is a constant.

Here and in the following we use the standard notation

$$\|f\|_q \equiv \begin{cases} \left(\int_\Omega |f|^q\right)^{1/q} & \text{if } 1 \le q < \infty\,, \\ \text{ess sup}|f(x)| & \text{if } q = \infty\,. \end{cases} \tag{2.15}$$

From $(2.10)_1$, by an integration by parts we find

$$\mathcal{N}_{1i} = \int_\Omega [(\mathbf{u}+\mathbf{w}) \cdot \operatorname{grad} \mathbf{H}^{(i)} \cdot (\mathbf{v}+\mathbf{U}) + \mathbf{v}_S \cdot \operatorname{grad} \mathbf{H}^{(i)} \cdot (\mathbf{u}+\mathbf{w})]\,. \tag{2.16}$$

For the sake of simplicity, we take (without loss) $V_0 = V$, so that $|\mathbf{U}| = 1$. From well-known results on the Stokes problem[10] we then find

$$\|\mathbf{v}_S\|_\infty \le c\,, \tag{2.17}$$

where c is a positive constant depending only on $\mathcal{B}$.

Assume now that there are Re_0, $\lambda_0 > 0$ such that for all $0 < \mathrm{Re} < \mathrm{Re}_0$, and $0 < \lambda < \lambda_0$ the following conditions hold

(H1) $\|\mathbf{v}+\mathbf{U}\|_\infty \le c_1$,
(H2) $\|\mathbf{u}\|_{q_1} \le c_2\lambda^{\beta_1}$, for *some* $q_1 \in (1,\infty)$, $\beta_1 > 0$,
(H3) $\|\mathbf{w}\|_{q_2} \le c_2\,\mathrm{Re}^{\gamma_1}$, for *some* $q_2 \in (1,\infty)$, $\gamma_1 > 0$,

where c_1, c_2, c_3 are (positive) constants depending only (at most) on $\mathcal{B}$, Re_0, λ_0 and q. Then, using Hölder's inequality and (2.14) in (2.16), we find

$$|\boldsymbol{\mathcal{N}}_1| \le c_4(\mathrm{Re}^{\gamma_1} + \lambda^{\beta_1})\,,$$

with a constant c_4 independent of Re and λ.

Likewise, *assume* that for all $0 < \mathrm{Re} < \mathrm{Re}_0$, and $0 < \lambda < \lambda_0$ the following conditions hold

(H4) $\|\mathbf{S}(\mathbf{v}) - \mathbf{S}(\mathbf{v}_{NS})\|_{q_3} \le c_2'\lambda^{\beta_2}$, for *some* $q_3 \in (1,\infty)$, $\beta_2 > 0$,
(H5) $\|\mathbf{S}(\mathbf{v}_{NS}) - \mathbf{S}(\mathbf{v}_S)\|_{q_4} \le c_3'\mathrm{Re}^{\gamma_2}$, for *some* $q_4 \in (1,\infty)$, $\gamma_2 > 0$,

with c_2', c_3' independent of Re, λ. Then, using again Hölder's inequality and (2.14) in $(2.10)_2$, we find

$$|\boldsymbol{\mathcal{N}}_2| \le c_4'(\mathrm{Re}^{\gamma_2} + \lambda^{\beta_2})\,.$$

The results just described are summarized in the following.

Lemma 2.1. *Let $\mathcal{B}$ be a homogeneous body of revolution with fore-and-aft symmetry. Assume that conditions* (H1)–(H5) *hold. Then, there are positive* Re_0 *and* λ_0 *such that for all* $0 < \mathrm{Re} \le \mathrm{Re}_0$, *and* $0 < \lambda \le \lambda_0$ *the total torque* (2.2) *exerted by the liquid $\mathcal{L}$ on $\mathcal{B}$ is given by*

$$\boldsymbol{\mathcal{M}} = \mathrm{Re}\boldsymbol{\mathcal{M}}^{0,I} + \lambda\boldsymbol{\mathcal{M}}^{0,NN} + \boldsymbol{\mathcal{N}}\,,$$

where $\boldsymbol{\mathcal{M}}^{0,I}$ and $\boldsymbol{\mathcal{M}}^{0,NN}$ are defined in (2.8), *while*

$$|\boldsymbol{\mathcal{N}}| \le C(\mathrm{Re}^{1+\gamma} + \lambda^{1+\beta})\,,$$

with C, γ and β positive constants independent of Re *and* λ.

3. The Torque for a Second-Order Fluid

The aim of this section is to show that conditions (H1)–(H5) of Lemma 2.1 are indeed satisfied for a second-order fluid. We recall that, in this case, the "extra-stress" $\mathbf{S}$ can be written as

$$\mathbf{S} = 2(\mathbf{A}_1 + 2\varepsilon\mathbf{D}^2)\,,$$

where $\mathbf{A}_1$ is the Rivlin–Ericksen tensor which in the steady case takes form

$$\mathbf{A}_1 = \mathbf{v}\cdot\operatorname{grad}\mathbf{D} + \mathbf{D}\cdot(\operatorname{grad}\mathbf{v})^{\mathrm{T}} + \operatorname{grad}\mathbf{v}\cdot\mathbf{D}\,.$$

Moreover, $\lambda \equiv \mathrm{We}$, where

$$\mathrm{We} = \frac{|\alpha_1|V_0}{d\mu} \quad (\text{Weissenberg number})\,,$$

$\varepsilon = \alpha_2/|\alpha_1|$, and α_1, α_2 are related to the normal stress coefficients Ψ_1 and Ψ_2 by the formulas $\alpha_1 = -\frac{1}{2}\Psi_1$, $\alpha_2 = \Psi_1 + \Psi_2$ (see Chap. 17 of Ref. 16). The equations of motion (2.1) then become

$$\left.\begin{aligned} &\mathrm{Re}\,\mathbf{v}\cdot\operatorname{grad}\mathbf{v} = \operatorname{div}(-p\mathbf{I} + 2\mathbf{D} + 2\mathrm{We}(\mathbf{A}_1 + 2\varepsilon\mathbf{D}^2))\\ &\operatorname{div}\mathbf{v} = 0\end{aligned}\right\}\text{ in }\Omega$$
$$\mathbf{v} = \mathbf{0}\quad\text{at }\Sigma\,, \tag{3.1}$$
$$\lim_{|x|\to\infty}\mathbf{v}(x) = -\mathbf{U}\,.$$

In the Navier–Stokes case, i.e. We $= 0$, the above problem specializes to the following one

$$\left.\begin{aligned} &\mathrm{Re}\,\mathbf{v}_{NS}\cdot\operatorname{grad}\mathbf{v}_{NS} = \operatorname{div}(-p_{NS}\mathbf{I} + 2\mathbf{D})\\ &\operatorname{div}\mathbf{v}_{NS} = 0\end{aligned}\right\}\text{ in }\Omega$$
$$\mathbf{v}_{NS} = \mathbf{0}\quad\text{at }\Sigma\,, \tag{3.2}$$
$$\lim_{|x|\to\infty}\mathbf{v}_{NS}(x) = -\mathbf{U}\,.$$

Finally, by taking Re $= 0$, this problem reduces, in turn, to the Stokes problem

$$\left.\begin{aligned} &\operatorname{div}(-p_S\mathbf{I} + 2\mathbf{D}) = \mathbf{0}\\ &\operatorname{div}\mathbf{v}_S = 0\end{aligned}\right\}\text{ in }\Omega$$
$$\mathbf{v}_S = \mathbf{0}\quad\text{at }\Sigma\,, \tag{3.3}$$
$$\lim_{|x|\to\infty}\mathbf{v}_S(x) = -\mathbf{U}\,.$$

In what follows, we need to assume some regularity on Ω. We, therefore, *suppose throughout that Ω is of class C^3.*

The key results of this section are collected in Theorems 3.1 and 3.2, while the main result is stated in Theorem 3.3. The proof of Thorems 3.1 and 3.2 is rather technical, and it will be given in Appendix A.

Theorem 3.1. *There exist positive numbers* $\mathrm{Re}_0 = \mathrm{Re}_0(\Omega, \varepsilon)$, $C_1 = C_1(\Omega, \mathrm{Re}_0, q)$ *and* $C_2 = C_2(\Omega, \mathrm{Re}_0, \varepsilon, q)$ *such that for any* $0 < \mathrm{Re} \le \mathrm{Re}_0$, *and* $1 < q < 3/2$ *we have*

(i) $\|\mathbf{v}_{NS} - \mathbf{v}_S\|_{\frac{3q}{3-2q}} \le C_1 \mathrm{Re}^{1-\eta}$,
(ii) $\|\mathbf{S}(\mathbf{v}_{NS}) - \mathbf{S}(\mathbf{v}_S)\|_q \le C_2 \mathrm{Re}^{1-\eta}$,

where η *can be taken arbitrarily close to zero, by choosing* q *arbitrarily close to* $3/2$ $(C_1, C_2 \to \infty$ *as* $q \to 3/2)$.

For a given $C > 0$, we shall say that a solution $\mathbf{v}, p$ to (2.1) *belongs to the class* $\mathcal{C}_C$ if and only if

$$\mathrm{Re}^{\frac{1}{2}} \|\mathbf{v} + \mathbf{U}\|_{\frac{2q}{2-q}} + \mathrm{Re}^{\frac{1}{4}} \|\mathrm{grad}\,\mathbf{v}\|_{\frac{4q}{4-q}} + \|D^2\mathbf{v}\|_{1,q} + \|D^2\mathbf{v}\|_{1,t} + \|\mathrm{grad}\,p\|_q + \|\mathrm{grad}\,p\|_t \le C\,.$$

We have the following.

Theorem 3.2. *Let* $\mathbf{v}, p \in \mathcal{C}_C$ *for some* $C > 0$. *Then, there exist positive numbers* $\mathrm{We}_0 = \mathrm{We}_0(\Omega, \varepsilon, C)$, $\mathrm{Re}_0 = \mathrm{Re}_0(\Omega, \varepsilon, C)$, *and* $C_3 = C_3(\Omega, \mathrm{We}_0, \mathrm{Re}_0, \varepsilon, q)$ *such that for any* $0 < \mathrm{Re} \le \mathrm{Re}_0$, $0 < \mathrm{We} \le \mathrm{We}_0$, *and* $1 < q < 3/2$ *we have*

(i) $\|\mathbf{v} + \mathbf{U}\|_\infty \le C_3$,
(ii) $\|\mathbf{v} - \mathbf{v}_{NS}\|_{\frac{3q}{3-2q}} \le C_3 \mathrm{We}$,
(iii) $\|\mathbf{S}(\mathbf{v}) - \mathbf{S}(\mathbf{v}_{NS})\|_q \le C_3 \mathrm{We}$.

From Lemma 2.1, Theorems 3.1 and 3.2 we immediately obtain the main result of this section.

Theorem 3.3. *Let* $\mathcal{B}$ *be a homogeneous body of revolution with fore-and-aft symmetry moving in a second-order liquid* $\mathcal{L}$ *by constant translational motion. Let* $\mathbf{v}, p \in \mathcal{C}_C$, *some* $C > 0$. *Then, there are positive* Re_0 *and* We_0 *depending on* $\mathcal{B}$, ε *and* C, *such that for all* $0 < \mathrm{Re} \le \mathrm{Re}_0$, *and* $0 < \mathrm{We} \le \mathrm{We}_0$ *the total torque* (2.2) *exerted by* $\mathcal{L}$ *on* $\mathcal{B}$ *is given by*

$$\boldsymbol{\mathcal{M}} = \mathrm{Re}\, \boldsymbol{\mathcal{M}}^{0,I} + \mathrm{We}\, \boldsymbol{\mathcal{M}}^{0,NN} + \boldsymbol{\mathcal{N}}\,, \tag{3.4}$$

with $\boldsymbol{\mathcal{M}}^{0,I}$ *and* $\boldsymbol{\mathcal{M}}^{0,NN}$ *defined in* (2.8), *while*

$$|\boldsymbol{\mathcal{N}}| \le K(\mathrm{Re}^{2-\eta} + \mathrm{We}^2)\,,$$

where K *and* η *are positive constants independent of* Re *and* We, *and where* η *can be taken arbitrarily close to zero* $(K \to \infty$ *as* $\eta \to 0)$.

4. On the Orientation of a Body of Revolution with Fore-and-Aft Symmetry Falling in a Second-Order Fluid at Small and Nonzero Re and We

Let $\mathcal{B}$ be a body of revolution around an axis a, with fore-and-aft symmetry with respect to a plane Π orthogonal to a. The objective of this section is to study the orientations of $\mathcal{B}$ falling by a translational motion with velocity $\mathbf{U}$ in a second-order liquid, under the action of acceleration of gravity $\mathbf{g}$, at small but nonzero Re and We.

Specifically, in the first part, we shall show that $\mathcal{B}$ has (at least) two possible ways of orienting itself, namely, with a either parallel or perpendicular to $\mathbf{g}$. Successively, using the results of Theorem 3.3 we show that, provided that the components $\mathcal{M}_3^{0,I}$ and $\mathcal{M}_3^{0,NN}$ (say) of $\boldsymbol{\mathcal{M}}^{0,I}$ and $\boldsymbol{\mathcal{M}}^{0,NN}$, respectively, in the plane orthogonal to $\mathbf{U}$ and $\mathbf{g}$ are not *both* zero, the mentioned orientations are *the only possible ones*. Finally, again using (3.4), we perform a "quasi-steady" stability analysis that shows that one or the other orientation is stable, depending on the competition between inertia and visco-elasticity, and on the sign of $\mathcal{M}_3^{0,I}$ and $\mathcal{M}_3^{0,NN}$.

In the next two sections we shall specialize these results to the case when $\mathcal{B}$ is a prolate spheroid and compare them with the experimental observations of Liu and Joseph.[21]

We recall that a body $\mathcal{B}$ moving under the action of gravity in a quiescent liquid $\mathcal{L}$ filling the whole space, is said to undergo a *free steady translational fall* if and only if there are $\mathbf{v}$, p, $\mathbf{U}$ and $\mathbf{g}$ satisfying the following problem,[27,30]

$$\left.\begin{aligned} &\operatorname{div}\mathbf{T} = \mathrm{Re}\,\mathbf{v}\cdot\operatorname{grad}\mathbf{v} + \mathbf{g}\\ &\operatorname{div}\mathbf{v} = 0\end{aligned}\right\}\ \text{in } \Omega$$

$$\begin{gathered}\mathbf{v} = \mathbf{0} \quad \text{on } \Sigma\\ \lim_{|x|\to\infty}\mathbf{v}(x) = -\mathbf{U}\\ \int_\Sigma \mathbf{T}(\mathbf{v},p)\cdot\mathbf{n} = m\mathbf{g}\\ \int_\Sigma \mathbf{x}\times\mathbf{T}(\mathbf{v},p)\cdot\mathbf{n} = 0\,.\end{gathered} \tag{4.1}$$

Definition 4.1. We shall say that a homogeneous body $\mathcal{B}$ is *symmetric* around the axis $a \equiv x_1$ (say), if and only if:

$$(x_1,x_2,x_3)\in\Sigma \Longrightarrow \begin{cases}(x_1,-x_2,x_3)\in\Sigma\,,\\ (x_1,x_2,-x_3)\in\Sigma\,.\end{cases}$$

One can show that every symmetric body can perform a steady translational fall, provided Re and We are not too large. Even though this fact seems very intuitive, its rigorous proof is rather technical and we will postpone it until Appendix B. We have:

Theorem 4.1. *Let $\mathcal{B}$ be a symmetric body around the axis a. Then, for $\mathbf{g}$ directed along a, there are $\mathrm{Re}_0, \mathrm{We}_0, C > 0$ depending only on $\mathcal{B}$ and ε, such that for any*

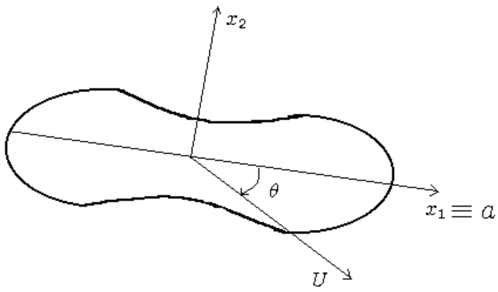


Fig. 3. Choice of the axes.

$0 \leq \mathrm{Re} \leq \mathrm{Re}_0$ *and* $0 \leq \mathrm{We} \leq \mathrm{We}_0$ *problem* (4.1) *has at least one solution* $(\mathbf{v}, p, \mathbf{U}, \mathbf{g})$ *with* $(\mathbf{v}, p) \in \mathcal{C}_C$. *The translational velocity* $\mathbf{U}$ *is also parallel to* $\mathbf{g}$ *with* $\mathbf{U} \cdot \mathbf{g} > 0$.

Moreover, if $(\mathbf{v}_1, p_1, \mathbf{U}, \mathbf{g}_1)$ *is any other solution to* (3.1) *with* $(\mathbf{v}_1, p_1) \in \mathcal{C}_C$, *then* $\mathbf{v}_1 = \mathbf{v}$, $p_1 = p$, *and* $\mathbf{g}_1 = \mathbf{g}$.

Since bodies of revolution around a, possessing fore-and-aft symmetry with respect to Π are symmetric (in the sense of Definition 4.1) around a and around any other axis belonging to Π, from Theorem 4.1 we deduce the following general result.

Theorem 4.2. *A homogeneous body of revolution around an axis* a, *possessing fore-and-aft symmetry can execute at small Reynolds and Weissenberg numbers at least two types of translational steady falls, determined by the following directions of* $\mathbf{g}$:

(a) $\mathbf{g}$ *is parallel to* a;
(b) $\mathbf{g}$ *is orthogonal to* a.

In both cases, $\mathbf{g}$ *is parallel to* $\mathbf{U}$, *with* $\mathbf{U} \cdot \mathbf{g} > 0$.[d]

Our next objective is to show that, at first order in Re and We, these are the only possible translational falls. In other words, the only possible orientations for $\mathcal{B}$ are with a either parallel or perpendicular to $\mathbf{g}$. A fundamental role in proving this result is played by the evaluation of the torque furnished in Theorem 3.3.

Without loss of generality, we take the x_1-axis of a frame attached to $\mathcal{B}$ coinciding with the axis of revolution a of $\mathcal{B}$, and assume the translational velocity $\mathbf{U}$ contained in the plane x_1, x_2; see Fig. 3. The Stokes velocity fields $\mathbf{v}_S$ solutions to (3.3), can then be expressed as

$$\mathbf{v}_S = U_1 \mathbf{h}^{(1)} + U_2 \mathbf{h}^{(2)} , \tag{4.2}$$

[d]We assume that the mass of the body minus the mass of the displaced liquid (effective mass) is positive, as appropriate for sedimentation.

where, for $i = 1, 2$,

$$\left.\begin{array}{l}\operatorname{div}\mathbf{T}_N(\mathbf{h}^{(i)}, p^{(i)}) = \mathbf{0}\\ \operatorname{div}\mathbf{h}^{(i)} = 0\end{array}\right\}\text{ in }\Omega$$
$$\mathbf{h}^{(i)} = \mathbf{0} \quad \text{at } \Sigma \tag{4.3}$$
$$\lim_{|x|\to\infty} \mathbf{h}^{(i)} = -\mathbf{e}_i\,.$$

We shall next introduce some symmetry classes. To this end we denote by $\mathcal{P}_i$, $i = 2, 3$, the following operators:

$$\begin{aligned}\mathcal{P}_1 f(x_1, x_2, x_3) &= f(-x_1, x_2, x_3)\,,\\ \mathcal{P}_2 f(x_1, x_2, x_3) &= f(x_1, -x_2, x_3)\,,\\ \mathcal{P}_3 f(x_1, x_2, x_3) &= f(x_1, x_2, -x_3)\,.\end{aligned} \tag{4.4}$$

We shall say that a vector field $\mathbf{w}$ *belongs to the class* $\mathcal{C}_1$ if and only if

$$w_1 = \mathcal{P}_2 w_1 = \mathcal{P}_3 w_1\,, \quad w_2 = -\mathcal{P}_2 w_2 = \mathcal{P}_3 w_2\,, \quad w_3 = \mathcal{P}_2 w_3 = -\mathcal{P}_3 w_3\,. \tag{4.5}$$

Likewise, $\mathbf{w} \in \mathcal{C}_2$ if and only if

$$w_1 = -\mathcal{P}_1 w_1 = \mathcal{P}_3 w_1\,, \quad w_2 = \mathcal{P}_1 w_2 = \mathcal{P}_3 w_2\,, \quad w_3 = \mathcal{P}_1 w_3 = -\mathcal{P}_3 w_3\,, \tag{4.6}$$

$\mathbf{w} \in \mathcal{C}_3$ if and only if

$$w_1 = -\mathcal{P}_1 w_1 = -\mathcal{P}_2 w_1\,, \quad w_2 = \mathcal{P}_1 w_2 = \mathcal{P}_2 w_2\,, \quad w_3 = \mathcal{P}_1 w_3 = -\mathcal{P}_2 w_3\,, \tag{4.7}$$

$\mathbf{w} \in \mathcal{C}_4$ if and only if

$$w_1 = \mathcal{P}_1 w_1 = \mathcal{P}_2 w_1\,, \quad w_2 = -\mathcal{P}_1 w_2 = -\mathcal{P}_2 w_2\,, \quad w_3 = -\mathcal{P}_1 w_3 = \mathcal{P}_2 w_3 \tag{4.8}$$

and finally, $\mathbf{w} \in \mathcal{C}_5$ if and only if

$$w_1 = \mathcal{P}_1 w_1 = -\mathcal{P}_2 w_1\,, \quad w_2 = -\mathcal{P}_1 w_2 = \mathcal{P}_2 w_2\,, \quad w_3 = -\mathcal{P}_1 w_3 = -\mathcal{P}_2 w_3\,. \tag{4.9}$$

Thus, using the symmetry properties of $\mathcal{B}$ and the uniqueness of Stokes problems (2.3) and (4.3), it is easy to show that $\mathbf{h}^{(i)}$, $i = 1, 2$, and $\mathbf{H}^{(j)}$, $j = 1, 2, 3$, satisfy the following conditions:

$$\begin{gathered}\mathbf{h}^{(1)} \in \mathcal{C}_1\,, \quad \mathbf{h}^{(2)} \in \mathcal{C}_2\\ \mathbf{H}^{(1)} \in \mathcal{C}_3\,, \quad \mathbf{H}^{(2)} \in \mathcal{C}_4\,, \quad \mathbf{H}^{(3)} \in \mathcal{C}_5\,.\end{gathered} \tag{4.10}$$

Using (4.5)–(4.9) along with (4.10), from (2.8) and (4.2) one shows directly that

$$\mathcal{M}_1^{0,I} = \mathcal{M}_2^{0,I} = \mathcal{M}_1^{0,NN} = \mathcal{M}_2^{0,NN} = 0$$

and that

$$\mathcal{M}_3^{0,I} = U_1 U_2 \mathcal{G}_{\mathcal{I}}\,, \quad \mathcal{M}_3^{0,NN} = U_1 U_2 \mathcal{G}_{\mathcal{V}}^{(\varepsilon)}\,, \tag{4.11}$$

where

$$\mathcal{G}_{\mathcal{I}} = -\int_\Omega (\mathbf{h}^{(1)} \cdot \operatorname{grad}\mathbf{h}^{(2)} + \mathbf{h}^{(2)} \cdot \operatorname{grad}\mathbf{h}^{(1)}) \cdot \mathbf{H}^{(3)} \tag{4.12}$$

and

$$\mathcal{G}_{\mathcal{V}}^{(\varepsilon)} = -\int_{\Omega} (\mathbf{h}^{(1)} \cdot \operatorname{grad} \mathbf{D}(\mathbf{h}^{(2)}) + (\operatorname{grad} \mathbf{h}^{(1)})^{\mathrm{T}} \cdot \mathbf{D}(\mathbf{h}^{(2)}) + \mathbf{D}(\mathbf{h}^{(1)}) \cdot \operatorname{grad} \mathbf{h}^{(2)} + \mathbf{h}^{(2)} \cdot \operatorname{grad} \mathbf{D}(\mathbf{h}^{(1)}) + (\operatorname{grad} \mathbf{h}^{(2)})^{\mathrm{T}} \cdot \mathbf{D}(\mathbf{h}^{(1)}) + \mathbf{D}(\mathbf{h}^{(2)}) \cdot \operatorname{grad} \mathbf{h}^{(1)} + 2\varepsilon \mathbf{D}(\mathbf{h}^{(1)}) \cdot \mathbf{D}(\mathbf{h}^{(2)})) : \mathbf{D}(\mathbf{H}^{(3)}) . \tag{4.13}$$

Clearly, the quantities $\mathcal{G}_{\mathcal{I}}$ and $\mathcal{G}_{\mathcal{V}}^{(\varepsilon)}$ (for fixed ε) depend only on the geometric properties of $\mathcal{B}$, such as size or shape, but are otherwise independent of the orientation of $\mathcal{B}$ and of the properties of the liquid. We call $\mathcal{G}_{\mathcal{I}}$ and $\mathcal{G}_{\mathcal{V}}^{(\varepsilon)}$ the *inertial torque coefficient* and *viscoelastic torque coefficient*, respectively, and set

$$\mathcal{G} = \operatorname{Re} \mathcal{G}_{\mathcal{I}} + \operatorname{We} \mathcal{G}_{\mathcal{V}}^{(\varepsilon)} . \tag{4.14}$$

Therefore, at first order in Re and We, from Theorem 3.3 we obtain that the torque $\boldsymbol{\mathcal{M}}$ acting on $\mathcal{B}$ is given by

$$\boldsymbol{\mathcal{M}} = \mathcal{G} U_1 U_2 \mathbf{e}_3 . \tag{4.15}$$

In the case of a steady fall, the torque must vanish (see $(4.1)_6$), and from (4.15) we deduce that, provided $\mathcal{G}$ is not zero, this can happen only if $\mathbf{U}$ is either directed along the axis of revolution a of $\mathcal{B}$ or it is perpendicular to it. From Theorem 4.1 it then follows that $\mathbf{U}$ has the same orientation as $\mathbf{g}$ and so we conclude that *provided $\mathcal{G}$ is not zero, the only possible orientations of $\mathcal{B}$ at first order in the Reynolds and Weissenberg numbers are with a either parallel or perpendicular to* $\mathbf{g}$.

Let us briefly analyze the possibility of having $\mathcal{G} = 0$. Strictly speaking, this means that the torque at first order is zero and, consequently, every orientation is possible at first order. However, for a given body $\mathcal{B}$ possessing *both* torque coefficients nonzero, as in the case of a prolate spheroid (see the next section), the condition $\mathcal{G} = 0$ is verified only for those values of Re and We belonging to the straight line passing through zero and with angular coefficient $-\mathcal{G}_{\mathcal{I}}/\mathcal{G}_{\mathcal{V}}^{(\varepsilon)}$ (see Fig. 9, for the case of a prolate spheroid). Therefore, in such a case, one can make $\mathcal{G} \neq 0$ by slightly changing Re or We (or both).

We shall now consider the stability of such orientations, when $\mathcal{G} \neq 0$. Since $U_1 = \cos\theta$, $U_2 = -\sin\theta$[e] (see Fig. 1), Eq. (4.15) can also be written as follows:

$$\boldsymbol{\mathcal{M}} = -\mathcal{G} \sin\theta \cos\theta \mathbf{e}_3 . \tag{4.16}$$

Thus, if we limit ourselves to perturbations in the form of infinitesimal disorientations of the type $\delta\theta e_3$, for a configuration to be stable [respectively, unstable] the variation of $\boldsymbol{\mathcal{M}}(\theta)$ from its value at the equilibrium configuration, should have a sign opposite to $\delta\theta$ [respectively, the same sign]. Therefore, denoting by θ_0 the

[e]Recall that $|\mathbf{U}| = 1$.

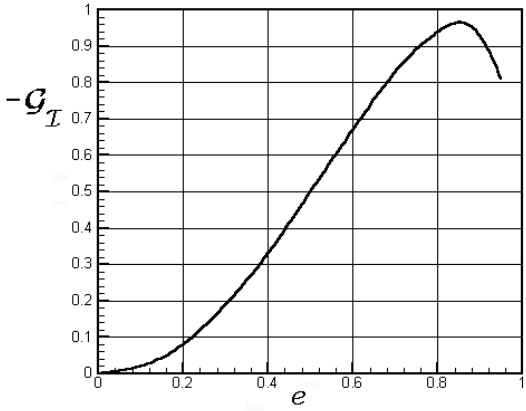


Fig. 4. Absolute value of the inertial torque coefficient as a function of e.

equilbrium configuration (i.e. θ_0 is either 0 or $\pi/2$), we have

$$\left.\frac{d(\boldsymbol{\mathcal{M}}\cdot\mathbf{e}_3)}{d\theta}\right|_{\theta=\theta_0} < 0 \Longrightarrow \text{stability}\,,$$

$$\left.\frac{d(\boldsymbol{\mathcal{M}}\cdot\mathbf{e}_3)}{d\theta}\right|_{\theta=\theta_0} > 0 \Longrightarrow \text{instability}\,.$$

Consequently, taking into account (4.14), we obtain

$$\theta = 0\begin{cases}\text{stable if} & \operatorname{Re}\mathcal{G}_{\mathcal{I}} > -\operatorname{We}\mathcal{G}_{\mathcal{V}}^{(\varepsilon)}\,,\\ \text{unstable if} & \operatorname{Re}\mathcal{G}_{\mathcal{I}} < -\operatorname{We}\mathcal{G}_{\mathcal{V}}^{(\varepsilon)}\,,\end{cases}$$

$$\theta = \frac{\pi}{2}\begin{cases}\text{stable if} & \operatorname{Re}\mathcal{G}_{\mathcal{I}} < -\operatorname{We}\mathcal{G}_{\mathcal{V}}^{(\varepsilon)}\,,\\ \text{unstable if} & \operatorname{Re}\mathcal{G}_{\mathcal{I}} > -\operatorname{We}\mathcal{G}_{\mathcal{V}}^{(\varepsilon)}\,.\end{cases}$$

From this we see that, perhaps at odds with intuition, the competition between the inertial and viscoelastic torques does not produce an "intermediate" equilibrium configuration corresponding to an angle $\theta \neq 0, \pi/2$, as conjectured in Ref. 18. Rather, it is only responsible for the stability/instability of the configurations $\theta = 0, \pi/2$.

5. The Orientation of a Prolate Spheroid Falling in a Second-Order Liquid

This section aims at discussing the nature of the torque in the case when $\mathcal{B}$ is a prolate spheroid of eccentricity e. The evaluation of $\mathcal{G}_I$ was already performed in Ref. 13 and it was found that it is always negative for $e \in (0,1)$ and that it becomes zero for $e = 0, 1$. Variation of $-\mathcal{G}_I$ with e is given in Fig. 4. We shall next evaluate the viscoelastic torque coefficient $\mathcal{G}_{\mathcal{V}}^{(\varepsilon)}$. In the case $\varepsilon = 1$ this has been done analytically in Ref. 11. However, the general case $\varepsilon \neq 1$ seems to be much more complicated, and we have evaluated $\mathcal{G}_{\mathcal{V}}^{(\varepsilon)}$ numerically using the software Mathematica (Wolfram Inc.). The procedure used here follows the arguments presented in Ref. 13 for the purely Newtonian case. Integrating by parts several times in the integral in (4.12), we find the following expression for $\mathcal{G}_{\mathcal{V}}^{(\varepsilon)}$:

$$\begin{aligned}\mathcal{G}_{\mathcal{V}}^{(\varepsilon)} = \Bigg\{ \frac{3-\varepsilon}{2} \int_{\Sigma} \mathbf{x} \times \mathbf{n}\boldsymbol{\omega}^{(1)} \cdot \boldsymbol{\omega}^{(2)} + (\varepsilon - 1) \Bigg[\int_{\Omega} (3\mathbf{H} \cdot \operatorname{grad} \mathbf{h}^{(1)} \cdot \Delta \mathbf{h}^{(2)} \\ + 3\mathbf{H} \cdot \operatorname{grad} \mathbf{h}^{(2)} \cdot \Delta \mathbf{h}^{(1)} + \mathbf{H} \cdot (\operatorname{grad} \mathbf{h}^{(1)})^{\mathrm{T}} \cdot \Delta \mathbf{h}^{(2)} \\ + \mathbf{H} \cdot (\operatorname{grad} \mathbf{h}^{(2)})^{\mathrm{T}} \cdot \Delta \mathbf{h}^{(1)}) \Bigg] \Bigg\}, \end{aligned} \tag{5.1}$$

where, for simplicity, we set $\mathbf{H}^{(3)} = \mathbf{H}$, and where $\boldsymbol{\omega}_i = \operatorname{curl} \mathbf{h}^{(i)}$, $i = 1, 2$. Note that for $\varepsilon = 1$, the volume integral in (5.1) disappears and we revert back to the case considered in Ref. 11. The evaluation of the volume integral becomes simpler if we notice that (see (4.3))

$$\Delta \mathbf{h}^{(i)} = \operatorname{grad} p^{(i)}$$

for $i = 1, 2$. So the torque coefficient takes the following form

$$\begin{aligned}\mathcal{G}_{\mathcal{V}}^{(\varepsilon)} = \Bigg\{ \frac{3-\varepsilon}{2} \int_{\Sigma} \mathbf{x} \times \mathbf{n}\boldsymbol{\omega}^{(1)} \cdot \boldsymbol{\omega}^{(2)} + (\varepsilon - 1) \Bigg[\int_{\Omega} (3\mathbf{H} \cdot \operatorname{grad} \mathbf{h}^{(1)} \cdot \operatorname{grad} p^{(2)} \\ + 3\mathbf{H} \cdot \operatorname{grad} \mathbf{h}^{(2)} \cdot \operatorname{grad} p^{(1)} + \mathbf{H} \cdot (\operatorname{grad} \mathbf{h}^{(1)})^{\mathrm{T}} \cdot \operatorname{grad} p^{(2)} \\ + \mathbf{H} \cdot (\operatorname{grad} \mathbf{h}^{(2)})^{\mathrm{T}} \cdot \operatorname{grad} p^{(1)} \Bigg] \Bigg\}. \end{aligned} \tag{5.2}$$

All fields involved in the integral are explicitly known; see, e.g., Ref. 7. The value of ε is not well established. However, there seems to be reason to believe that $1.6 \le \varepsilon \le 2$.[2,21] A detailed discussion on the effect of this parameter is included at the end of the section.

In Cartesian components, the fields $\mathbf{h}^{(1)}$, $\mathbf{h}^{(2)}$, and the corresponding pressure fields, p_1 and p_2, are given by[7]

$$\begin{aligned}\mathbf{h}^{(1)} &= -\mathbf{e}_1 + 2\alpha_1 \mathbf{e}_1 B_{10} + \alpha_1 r \mathbf{e}_r \left(\frac{1}{R^2} - \frac{1}{R_1} \right) - \alpha_1 r^2 \mathbf{e}_1 B_{30} + 2\beta_1 \operatorname{grad} B_{11}, \\ \mathbf{h}^{(2)} &= -\mathbf{e}_2 + \alpha_2 \mathbf{e}_2 B_{10} + \alpha_2 x_2 \mathbf{e}_1 \left(\frac{1}{R^2} - \frac{1}{R_1} \right) - \alpha_2 r x_2 \mathbf{e}_r B_{30} - \beta_2 \operatorname{grad} B, \\ p_1 &= 2\alpha_1 \left(\frac{1}{R_1} - \frac{1}{R_2} \right), \quad p_2 = 2\alpha_2 \frac{x_2}{r^2} \left(\frac{x-e}{R_2} - \frac{x+e}{R_1} \right),\end{aligned}$$

while the field $\mathbf{H}$ is given by[7]

$$\begin{aligned}\mathbf{H} = ((\alpha_3 - \alpha_3')(2A_1 + A_3)x_2 + (\gamma_3' - \gamma_3)A_3 x_2)\mathbf{e}_1 + (2(\alpha_3 - \alpha_3')B_{31}x_2^2 \\ + 2(\gamma_3 - \gamma_3')B_{11})\mathbf{e}_2 + 2(\alpha_3 - \alpha_3')B_{31}x_2 x_3 \mathbf{e}_3 + 4(\beta_3 - \beta_3')\operatorname{grad}[x_2 A_2].\end{aligned}$$

In the above formulas we have set $\mathbf{e}_r = (x_2\mathbf{e}_2 + x_3\mathbf{e}_3)/r$ and

$$r = \sqrt{x_2^2 + x_3^2}\,, \quad R = \sqrt{x_1^2 + r^2}\,,$$

$$R_1 = \sqrt{(x_1+e)^2 + r^2}\,, \quad R_2 = \sqrt{(x_1-e)^2 + r^2}\,,$$

$$B = x_2\left(\frac{x_1-e}{r^2}R_1 - \frac{x_1+e}{r^2}R_2 + B_{10}\right),$$

$$B_{10} = \ln\frac{R_2 - x_1 + e}{R_1 - x_1 - e}\,, \quad B_{11} = R_2 - R_1 + B_{10}\,,$$

$$B_{30} = \frac{1}{r^2}\left(\frac{x_1+e}{R_2} - \frac{x_1-e}{R_1}\right), \quad B_{31} = \left(\frac{1}{R_2} - \frac{1}{R_1}\right) + x_1 B_{30}\,,$$

$$B_{32} = -e\left(\frac{1}{R_2} + \frac{1}{R_1}\right) + B_{10} + x_1 B_{31}\,, \quad B_{33} = -e^2\left(\frac{1}{R_2} - \frac{1}{R_1}\right) + 2B_{11} + x_1 B_{32}\,,$$

$$A_1 = x_1 B_{31} - B_{32}\,, \quad A_2 = e^2 B_{30} - B_{33}\,, \quad A_3 = e^2 B_{30} - B_{32}\,,$$

$$\alpha_1 = e^2\left[-2e + (1+e^2)\ln\frac{1+e}{1-e}\right]^{-1}, \quad \alpha_2 = 2e^2\left[2e + (3e^2-1)\ln\frac{1+e}{1-e}\right]^{-1},$$

$$\beta_1 = \frac{(1-e^2)\alpha_1}{2e^2}\,, \quad \beta_2 = \frac{(1-e^2)\alpha_2}{2e^2}\,,$$

$$\alpha_3 = \frac{4e^2}{(1-e^2)}\beta_3 = 2e^2\gamma_3\left[-2e + \ln\frac{1+e}{1-e}\right]\left[2e(2e^2-3) + 3(1-e^2)\ln\frac{1+e}{1-e}\right]^{-1},$$

$$\gamma_3 = (1-e^2)\left[-2e + (1+e^2)\ln\frac{1+e}{1-e}\right]^{-1},$$

$$\alpha_3' = \frac{4e^2}{(1-e^2)}\beta_3' = e^2\gamma_3'\left[-2e + (1-e^2)\ln\frac{1+e}{1-e}\right],$$

$$\times\left[2e(2e^2-3) + 3(1-e^2)\ln\frac{1+e}{1-e}\right]^{-1},$$

$$\gamma_3' = \frac{\gamma_3}{e^2-1}\,.$$

It is next observed that the calculation of $\mathcal{G}_{\mathcal{V}}^{(\varepsilon)}$ is considerably simpler in prolate-spheroidal coordinates (ζ, μ, θ) with the transformation from Cartesian coordinates given by

$$x_1 = e\mu\zeta\,, \quad x_2 = e\sqrt{\mu^2-1}\sqrt{1-\zeta^2}\cos\theta\,, \quad x_3 = e\sqrt{\mu^2-1}\sqrt{1-\zeta^2}\sin\theta\,,$$

and the scale factors

$$q_\mu = \frac{\sqrt{\mu^2-1}}{e\sqrt{\mu^2-\zeta^2}}\,, \quad q_\zeta = \frac{\sqrt{1-\zeta^2}}{e\sqrt{\mu^2-\zeta^2}}\,, \quad q_\theta = \frac{1}{e\sqrt{\mu^2-1}\sqrt{1-\zeta^2}}\,.$$

The components of $h^{(i)}$, $i = 1, 2, H, p_1$ and p_2 in these new coordinates are given by

$$h_\mu^{(1)} = \zeta u_1\,, \quad h_\zeta^{(1)} = u_2\,, \quad h_\theta^{(1)} = 0\,,$$
$$h_\mu^{(2)} = v_1 \cos\theta\,, \quad h_\zeta^{(2)} = v_2 \zeta \cos\theta\,, \quad h_\theta^{(2)} = v_3 \sin\theta\,,$$
$$H_\mu = H_1 \zeta \cos\theta\,, \quad H_\zeta = H_2 \cos\theta\,, \quad H_\theta = H_3 \zeta \sin\theta\,,$$
$$p_1 = \zeta P_1\,, \quad p_2 = P_2 \cos\theta\,,$$

with

$$u_1 = \frac{1}{\sqrt{\mu^2-1}\sqrt{\mu^2-\zeta^2}}\left(1 - 4\beta_1\mu + 2(\alpha_1+\beta_1)(\mu^2-1)\ln\frac{\mu+1}{\mu-1}\right)\,,$$

$$u_2 = \frac{\sqrt{1-\zeta^2}}{\sqrt{\mu^2-\zeta^2}}\left(-2\alpha_1 - 4\beta_1 - \mu + 2(\alpha_1+\beta_1)\mu\ln\frac{\mu+1}{\mu-1}\right)\,,$$

$$v_1 = \frac{\sqrt{1-\zeta^2}}{(\mu^2-1)\sqrt{\mu^2-\zeta^2}}\Bigg(-4\beta_2 + \mu + 2\beta_2\mu^2 - \mu^3 + 2\alpha_2(\mu^2-1)$$
$$+ (\alpha_2-\beta_2)\mu(\mu^2-1)\ln\frac{\mu+1}{\mu-1}\Bigg)\,,$$

$$v_2 = \frac{1}{\sqrt{\mu^2-1}\sqrt{\mu^2-\zeta^2}}\left(-1 - 2\beta_2\mu + \mu^2 - (\alpha_2-\beta_2)(\mu^2-1)\ln\frac{\mu+1}{\mu-1}\right)\,,$$

$$v_3 = \frac{1}{\mu^2-1}\left(-1 - 2\beta_2\mu + \mu^2 - (\alpha_2-\beta_2)(\mu^2-1)\ln\frac{\mu+1}{\mu-1}\right)\,,$$

$$H_1 = \frac{-e\sqrt{1-\zeta^2}}{(\mu^2-1)\sqrt{\mu^2-\zeta^2}}\Bigg[6(1-\mu^2)(\alpha_3' - \alpha a_3 + \gamma_3' - \gamma_3) + (48\mu^2-56)(\beta_3'-\beta_3)$$
$$+ (\mu^2-1)\ln\frac{\mu+1}{\mu-1}\{(3\mu^2-3)(\alpha_3-\alpha_3') + (1-3\mu^2)(\gamma_3-\gamma_3')$$
$$+ (12-24\mu^2)(\beta_3'-\beta_3)\}\Bigg]\,,$$

$$H_2 = \frac{e}{\sqrt{\mu^2-1}(\mu^2-\zeta^2)^{\frac{3}{2}}}\Bigg[2\zeta^2(\mu^2-1)\Bigg\{2(-1+\zeta^2)(\alpha_3-\alpha_3') + 2\zeta^2(\gamma_3'-\gamma_3)$$
$$+ 2\mu^2(\gamma_3-\gamma_3') - (\gamma_3-\gamma_3')\mu(\mu^2-\zeta^2)\ln\frac{\mu+1}{\mu-1}\Bigg\} + 4(\beta_3-\beta_3')(2\zeta^2-1)$$
$$\times\left\{4 - 6\mu^2 + 3\mu(\mu^2-1)\ln\frac{\mu+1}{\mu-1}\right\} - \frac{\mu(1-\zeta^2)}{(\mu^2-\zeta^2)}\Bigg\{2\mu(\alpha_3-\alpha_3')(2+\zeta^2-3\mu^2)$$
$$+ 2\mu(\gamma_3'-\gamma_3)(\zeta^2-\mu^2) + (3\alpha_3 - 3\alpha_3' + \gamma_3' - \gamma_3)(\mu^2-\zeta^2)(\mu^2-1)\ln\frac{\mu+1}{\mu-1}\Bigg\}\Bigg]\,,$$

$$H_3 = \frac{2e}{\mu^2 - 1}\left[(8 - 12\mu^2)(\beta_3 - \beta_3') + 2(\mu^2 - 1)(\gamma_3 - \gamma_3')\right.$$
$$\left. + (6\beta_3 - 6\beta_3' + \gamma_3' - \gamma_3)\mu(\mu^2 - 1)\ln\frac{\mu + 1}{\mu - 1}\right],$$

$$P_1 = \frac{-4\alpha_1}{e(\mu^2 - \zeta^2)},$$

$$P_2 = \frac{-4\alpha_2\mu\sqrt{1 - \zeta^2}}{e\sqrt{(\mu^2 - 1)}(\mu^2 - \zeta^2)}.$$

We write the components of $\mathbf{h}^{(1)}$, $\mathbf{h}^{(2)}$ and $\mathbf{H}$ in the above form in order to exploit the symmetry of the problem in the ζ and θ directions.

Next we perform the numerical evaluation of the integral in (5.2) with the software Mathematica (Wolfram Inc.).

Graphs of the variation of the viscoelastic torque coefficient $\mathcal{G}_{\mathcal{V}}^{(\varepsilon)}$ with eccentricity are shown in Figs. 6 and 7. They also depict the variation of $\mathcal{G}_{\mathcal{V}}^{(\varepsilon)}$ with the parameter ε. The essential profile of the curve stays remarkably consistent for each value of the parameter ε (see Fig. 6), changing slightly when $\varepsilon < 1$ (see Fig. 7). Also, $\mathcal{G}_{\mathcal{V}}^{(\varepsilon)}$ increases with increasing ε. It is also interesting to note in Fig. 7 that $\mathcal{G}_{\mathcal{V}}^{(\varepsilon)}$ is always positive for each e if ε is greater than approximately 1. As ε becomes less than one, the torque coefficient changes sign for e close to one. Let us analyze the two situations $\varepsilon \gtrsim 1$ and $\varepsilon \lesssim 1$ separately.

The case $\varepsilon \gtrsim 1$. In this case we have that $\mathcal{G}_{\mathcal{I}}$ and $\mathcal{G}_{\mathcal{V}}^{(\varepsilon)}$ *have opposite sign*. In view of the results of Sec. 2, this means that *for $\varepsilon \gtrsim 1$ the stable orientation of the prolate spheroid is with its major axis a perpendicular to the gravity* $\mathbf{g}$ *if* $-\mathrm{Re}\,\mathcal{G}_{\mathcal{I}} > \mathrm{We}\,\mathcal{G}_{\mathcal{V}}^{(\varepsilon)}$ (*inertia prevails on normal stresses*) *while the stable orientation is with a parallel to* $\mathbf{g}$ *if* $-\mathrm{Re}\,\mathcal{G}_{\mathcal{I}} < \mathrm{We}\,\mathcal{G}_{\mathcal{V}}^{(\varepsilon)}$ (*normal stresses prevail on inertia*). This result can be more easily stated in terms of the *elasticity number* $E = \mathrm{We}/\mathrm{Re}$. Specifically, introducing the critical elasticity number

$$E_{\mathrm{c}} = E_{\mathrm{c}}(e, \varepsilon) \equiv \frac{|\mathcal{G}_{\mathcal{I}}|}{\mathcal{G}_{\mathcal{V}}^{(\varepsilon)}},$$

we have that if $E < E_{\mathrm{c}}$ the liquid behaves as Newtonian, that is the orientation with a perpendicular to $\mathbf{g}$ is stable, while if $E > E_{\mathrm{c}}$, the other is stable. A plot of $E_{\mathrm{c}}(e, \varepsilon)$ versus e, for $\varepsilon = 1, 1.8$ is given in Fig. 5.

The case $\varepsilon \lesssim 1$. For values of eccentricities in the range $(0, \sim 0.9)$ the stability of the equilibrium configuration is the same as in the case $\varepsilon \gtrsim 1$. However, for very *slender spheroids* ($e \sim 1$) $\mathcal{G}_{\mathcal{V}}^{(\varepsilon)}$ *becomes negative*. Therefore, *if $\varepsilon \lesssim 1$, sedimenting slender spheroids will have $\mathcal{G}_{\mathcal{I}}$ and $\mathcal{G}_{\mathcal{V}}^{(\varepsilon)}$ acting in the same direction, and the configuration with a perpendicular to* $\mathbf{g}$ *is always stable, as in the case of a purely Newtonian fluid*. Since slender bodies in a viscoelastic fluid orient themselves with a parallel to $\mathbf{g}$,[19] our result confirms that the predicted value of a lower bound of

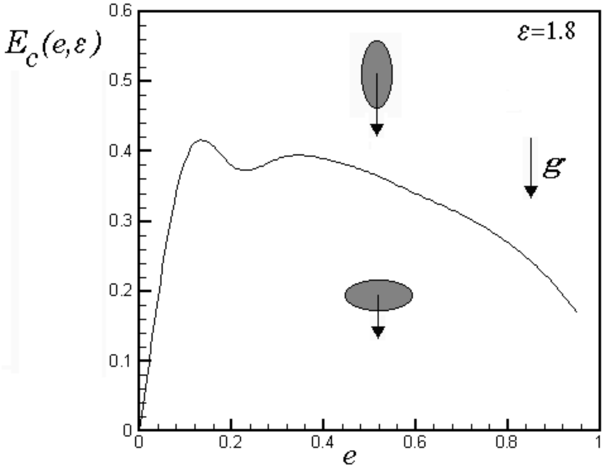


Fig. 5. Critical elasticity number E_c versus eccentricity e, for $\varepsilon = 1.8$. If $E > E_c$ the ellipsoid falls with its major axis a parallel to $\mathbf{g}$, while if $E < E_c$, the fall with a parallel to $\mathbf{g}$ is stable.

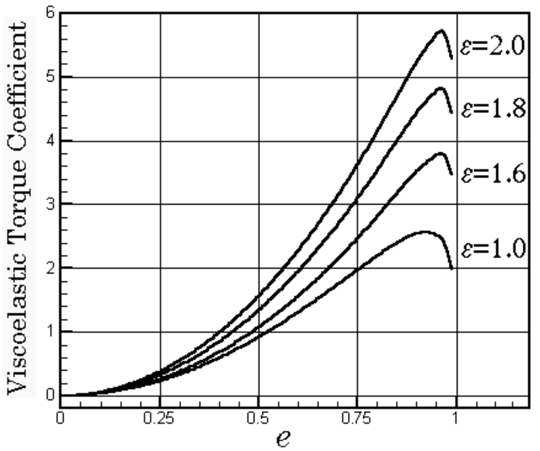


Fig. 6. Viscoelastic torque coefficient $\mathcal{G}_{\mathcal{V}}^{(\varepsilon)}$ vs. eccentricity e for different values of ε.

~ 1.6 for ε is appropriate. In Fig. 7 we plot $\mathcal{G}_{\mathcal{V}}^{(\varepsilon)}$ versus e for $\varepsilon = 0.7, 0.8$, since the dramatic turn to negative values is more prominent in these cases.

Another important feature is that the viscoelastic torque coefficient is several times larger than the absolute value of the inertial torque coefficient, mainly for eccentricities close to 1. Figure 8 compares the absolute value of the inertial torque coefficient to the viscoelastic torque coefficient for two different values of ε. We have chosen $\varepsilon = 1.8$ which is the value recommended in the experiments of Liu & Joseph,[21] see also Sec. 17.11 of Ref. 16, and $\varepsilon = 1$, that is the value for which the viscoelastic torque coefficient can be computed analytically.[11] The viscoelastic

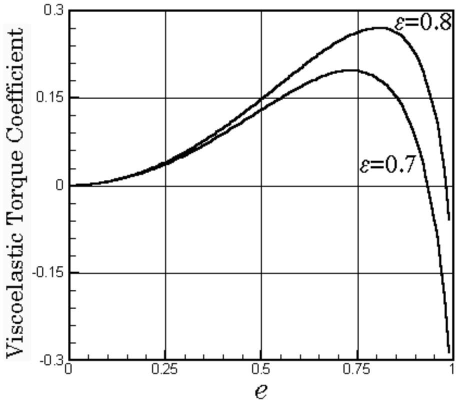


Fig. 7. Viscoelastic torque coefficient $\mathcal{G}_{\mathcal{V}}^{(\varepsilon)}$ versus e, for $\varepsilon = 0.7, 0.8$. The torque coefficient changes sign for e at approximately 0.9. Note also that the curves achieve their peaks at decreasing values of e as ε decreases.

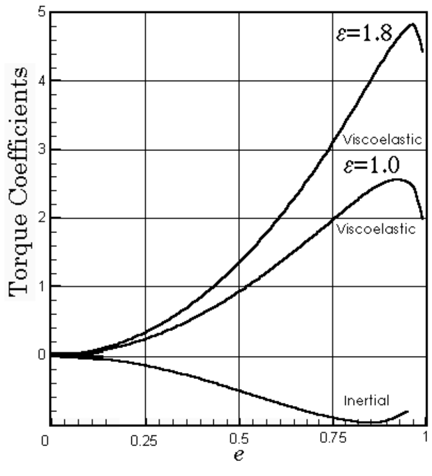


Fig. 8. Comparision of the inertial torque coefficient to the viscoelastic torque coefficient, for different values of ε. The viscoelastic torque coefficient is almost five times larger than the inertial one for e around 1 (slender ellipsoids) and $\varepsilon = 1.8$.

effects seem to outweigh the inertial ones. This helps explain the experimental observation which we shall discuss in the following section.

6. Comparison with Experiments

It is interesting to see how our theoretical results match with the experiments of Liu & Joseph[21] on particle sedimentation. Typically, particles considered by these authors are cylinders with round or flat ends. One of the main features of these studies is the observation of equilibrium configurations where the axis of the cylinders forms an angle with the horizontal, the *tilt angle* β_{tilt}, that may

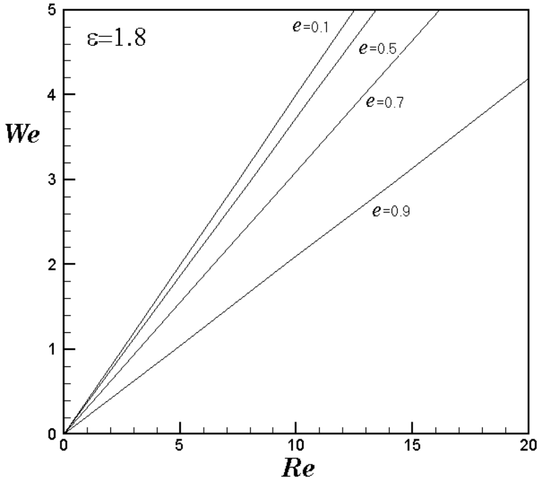


Fig. 9. Critical ratios of inertial versus viscoelastic torques for varying eccentricities e.

range *continuously* from 90° to 0°. These configurations are achieved by suitably varying the weight of the particles and the polymeric concentration of the liquid. Liu and Joseph suggest that the existence of the "intermediate" configuration, i.e. $0^\circ < \beta_{\mathrm{tilt}} < 90^\circ$, is due to the balance between inertia and viscoelastic torques. We have seen earlier that the second-order model only predicts $\beta_{\mathrm{tilt}} = 0^\circ, 90^\circ$.

Recall that in the equilibrium state, the balance of inertial to viscoelastic torques, at first order in Reynolds and Weissenberg numbers is represented by

$$(\mathrm{Re}\,\mathcal{G}_{\mathcal{I}} + \mathrm{We}\,\mathcal{G}_{\mathcal{V}}^{(\varepsilon)})\cos\theta\sin\theta = 0\,.$$

Introducing, for $e \neq 0, 1$, the ratio ρ of inertia to viscoelastic torque[f]:

$$\rho = \frac{\mathrm{Re}|\mathcal{G}_{\mathcal{I}}|}{\mathrm{We}\mathcal{G}_{\mathcal{V}}^{(\varepsilon)}}\,,$$

we then have a critical ratio

$$\rho_{\mathrm{c}} = 1\,. \tag{6.1}$$

When $\rho > \rho_{\mathrm{c}}$, then inertia dominates and the spheroid falls horizontally (i.e. $\beta_{\mathrm{tilt}} = 0^\circ$), while when $\rho < \rho_{\mathrm{c}}$, viscoelasticl effects dominate and the spheroid falls vertically (i.e. $\beta_{\mathrm{tilt}} = 90^\circ$). Figure 9 shows the critical curves for varying eccentricities e. The critcal curves are seen to be lines of varying slopes for the different eccentricities. Qualitatively, since $\mathcal{G}_{\mathcal{V}}^{(\varepsilon)}$ is much larger than $\mathcal{G}_{\mathcal{I}}$ (Fig. 8), varying the ratio of Re and We would determine the final orientation of the body. For the body to acquire the horizontal state, Re should far exceed We by an amount which can be read from Fig. 9.

[f]Recall that, for $e \neq 0, 1$, $\mathcal{G}_{\mathcal{I}}$ is negative, and $\mathcal{G}_{\mathcal{V}}^{(\varepsilon)}$ is positive, in the range of physically meaningful values of ε.

We shall now make a comparison of our results with experimental observations. Liu and Joseph have observations of the tilt angle for varying materials, and varying Re and We. It must be emphasized that the experiments we are referring to were performed using cylinders with flat and round edges whereas the results presented here are for ellipsoidal objects. It has been noted,[19,21] that the phenomenon of tilt angle is very sensitive to the geometry of the body. Hence we make this comparison with some apprehension. However, lack of sufficient experimental data leaves us with no other choice. Also, it must be noted that "eccentricities" of the cylinders used in the experiments have been approximated to between 0.85 and 0.92. Also, we take $\sqrt{1 - D^2/L^2}$ as a measure of the "eccentricity", where D is the diameter of the cylinder and L is its length. We set $L = 0.8$ inches. The drawback of this is that at eccentricities close to 1 the torques seem to drop rather rapidly. A small variation in the approximation of e may lead to a large error in the corresponding value of the torque. Therefore, more observations at lower eccentricities would render a more reliable comparison. This will be the object of future experimental work.

Figures 10 and 11 show how experimental observations match with our theoretical predictions. Comparisons have been shown for three different values of ε indicated on the plot. They expect to give an idea of how predictions of the experiment get better with increasing values of ε. The different symbols on each plot refer to the different materials used. ◆ represents brass, ▲ aluminum, ■ plastic and ◀ tin. The observed tilt angles are mentioned besides the plotted points. The dashed line indicates the critical ratio ρ_c. If the observations lie above the line, then the predicted tilt angle is $\beta_{\text{tilt}} = 0°$, otherwise $\beta_{\text{tilt}} = 90°$. It is seen from the figures above that the predictions seem to get better progressively, with increasing ε. For the case when $\varepsilon = 1.0, 1.6$ the ratio of the two torques is not quite large enough. However, at $\varepsilon = 1.8$ all the experimental data fall in the correct category of the graph. As mentioned earlier, the model fails to account for the tilt angle.

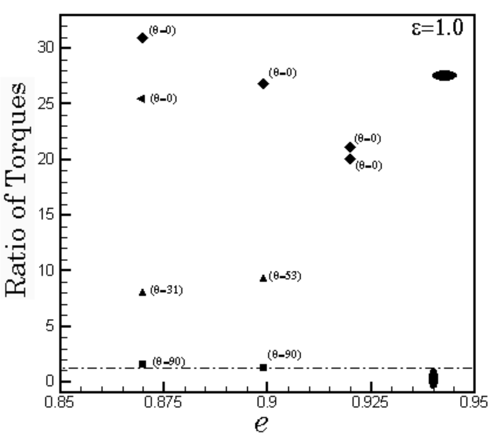


Fig. 10. Comparison with experimental data for $\varepsilon = 1$.

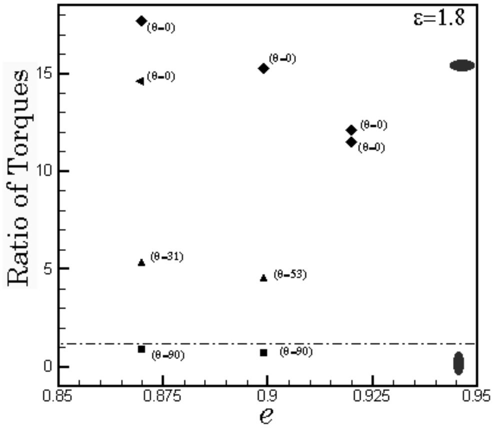


Fig. 11. Comparison with experimental data for $\varepsilon = 1.8$.

The two observed cases of tilt angle (namely of 53° and 31°) fall above the critical line for each off the cases. The calculations here seem to suggest, one more time, that $\varepsilon = 1.8$ better explains the experimental data than $\varepsilon = 1$. However, as is evident, a more complicated model is required to verify and explain the tilt angle phenomenon. This will be the object of future work.

Appendix A. Proof of Theorems 3.1 and 3.2

We begin to introduce some standard notation. As a rule, if X is a space of scalar functions, we shall use the same symbol X to denote the corresponding space of vector and tensor-valued functions. By $L^q(\Omega)$, $1 \le q \le \infty$, we denote the usual Lebesgue spaces with the associated norm (2.15). Moreover, $W^{m,q}(\Omega)$, $m \ge 0$, is the Sobolev space of functions that belong to $L^q(\Omega)$ together with their (generalized) derivatives up to the order m inclusive. Furthermore, by $W^{m-1/q,q}(\Sigma)$, we indicate the trace space at the boundary Σ of functions from $W^{m,q}(\Omega)$. Norms in $W^{m,q}(\Omega)$ and $W^{m-1/q,q}(\Sigma)$ will be denoted by $\|\cdot\|_{m,q}$ and $\|\cdot\|_{m-1/q,q(\Sigma)}$, respectively. Finally, we set

$$D^2 f = \frac{\partial^{l_i+l_j} f}{x_i^{l_i} \partial x_j^{l_j}}\,, \quad l_i + l_j = 2\,, \quad l_i, l_j = 0, 1\,, \quad i, j = 1, 2, 3\,.$$

In order to prove these theorems, we recall several preliminary results.

The first one concerns the Stokes problem and its proof is given in Theorems V.4.1 and V.5.1 of Ref. 10,

Lemma A.1.

$$\mathbf{F} \in L^q(\Omega)\,, \quad 1 < q < 3/2\,.$$

Then the problem

$$\left.\begin{aligned}\Delta\mathbf{u} &= \mathbf{F} + \operatorname{grad}\phi \\ \operatorname{div}\mathbf{u} &= 0\end{aligned}\right\} \textit{in } \Omega$$
$$\mathbf{u} = \mathbf{0} \quad \textit{on } \Sigma \tag{A.1}$$
$$\lim_{|x|\to\infty} \mathbf{u}(x) = 0$$

admits one and only one solution $\mathbf{u}, \phi$ *such that*

$$u \in L^{3q/(3-2q)}(\Omega)\,, \quad \operatorname{grad}\mathbf{u} \in L^{3q/(3-q)}(\Omega)\,, \quad D^2\mathbf{u} \in L^q(\Omega)$$
$$\phi \in L^{3q/(3-q)}(\Omega)\,, \quad \operatorname{grad}\phi \in L^q(\Omega)\,.$$

This solution satisfies the following estimate:

$$\|u\|_{3q\frac{3q}{3-2q}} + \|\operatorname{grad}\mathbf{u}\|_{\frac{3q}{3-q}} + \|D^2\mathbf{u}\|_q + \|\phi\|_{\frac{3q}{3-q}} + \|\operatorname{grad}\phi\|_q \le c_1\|\mathbf{F}\|_q\,,$$

where the positive constant c_1 *depends on* q *and* Ω.

The second one concerns the Oseen problem. Let $(\mathbf{h}, \pi)$ be a solution to the Oseen problem in Ω, i.e.

$$\begin{aligned}&-\Delta\mathbf{h} + \mathrm{Re}\frac{\partial\mathbf{h}}{\partial x_1} + \operatorname{grad}\pi = \mathbf{f} \\ &\operatorname{div}\mathbf{h} = 0\end{aligned} \tag{A.2}$$

with

$$\begin{aligned}&\mathbf{Z} = \mathbf{Z}_* \quad \text{at } \Sigma \\ &\lim_{|x|\to\infty} \mathbf{h}(x) = \mathbf{0}\,.\end{aligned} \tag{A.3}$$

We have

Lemma A.2. *Suppose that*

$$\mathbf{f} \in L^q(\Omega) \cup W^{1,t}(\Omega)\,, \quad 1 < q < \frac{3}{2}\,, \quad 1 < t < \infty\,, \quad \mathbf{Z}_* \in W^{3-\frac{1}{t},t(\Sigma)}\,.$$

Then, there exists a unique corresponding solution $(\mathbf{Z}, \pi)$ *to the Oseen problem* (A.2)–(A.3). *Moreover*

$$\begin{aligned}&a_1\|\mathbf{Z}\|_{\frac{2q}{2-q}} + a_2\|\operatorname{grad}\mathbf{Z}\|_{\frac{4q}{4-q}} + \|D^2\mathbf{Z}\|_q + \delta\|D^2\mathbf{Z}\|_{1,t} + \|\operatorname{grad}\pi\|_q + \delta\|\operatorname{grad}\pi\|_{1,t} \\ &\quad\le C(\|\mathbf{f}\|_q + \delta\|\mathbf{f}\|_{1,t} + \|\mathbf{Z}_*\|_{2-\frac{1}{q},q(\Sigma)} + \delta\|\mathbf{Z}_*\|_{3-\frac{1}{t},t(\Sigma)})\end{aligned} \tag{A.4}$$

with $a_1 = \min(1, \mathrm{Re}^{\frac{1}{2}})$, $a_2 = \min(1, \mathrm{Re}^{\frac{1}{4}})$, $\delta = 0$ *or* 1.

Proof. The part with $\delta = 0$ is proved in Chap. VII of Ref. 10, the other part can be easily deduced from results given there. See also Sec. III.5.3 of Ref. 26. □

We also need the following result on the steady transport equation, for whose proof we refer to Ref. 24; see also Refs. 12 and 26.

Lemma A.3. *Consider the following transport equation*

$$\mathbf{w} + \lambda \mathbf{u} \cdot \operatorname{grad} \mathbf{w} = \mathbf{f} \tag{A.5}$$

where λ *is a positive number*, $\mathbf{f} \in W^{1,q}(\Omega)$, $1 < q < \infty$, *and* $\mathbf{u}$ *is a solenoidal vector field in* Ω, *such that*

$$\mathbf{u} \in C^1(\bar{\Omega}), \quad \textit{and} \quad \mathbf{u} \cdot \mathbf{n} = 0 \quad \textit{at } \Sigma .$$

Then, if

$$\lambda \|\mathbf{u}\|_{C^1(\bar{\Omega})} < 1/2 ,$$

Eq. (A.3) *has one and only one solution* $\mathbf{w} \in W^{1,q}(\Omega)$. *Moreover, the solution satisfies the estimates*

$$\|\mathbf{w}\|_q \le \|\mathbf{f}\|_q ,$$

$$\|\operatorname{grad} \mathbf{w}\|_q \le 2 \|\operatorname{grad} \mathbf{f}\|_q .$$

The next two results regard the nonlinear problems (3.2) and (3.1). The following is a particular case of Lemma 2.2 proved in Ref. 13.

Lemma A.4. *Let* $\mathbf{v}_{NS}, p_{NS}$ *be a solution to* (2.2), *with* $\operatorname{grad} \mathbf{v}_{NS} \in L^2(\Omega)$, *and let* $1 < s < 3/2$. *There exists* $c_0 = c_0(\Omega, s) > 0$ *such that if* $\mathrm{Re} \le c_0$, *then* $\mathbf{v}_{NS}$ *satisfies the following estimates for all* $\sigma \in (3/2, +\infty)$.

$$\begin{aligned} &\|(\mathbf{v}_{NS} - \mathbf{U})(1 + |x|)\|_\infty + \|\operatorname{grad} \mathbf{v}_{NS}\|_\sigma \\ &\qquad + (\mathrm{Re})^{\frac{1}{4}} \|\operatorname{grad} \mathbf{v}_{NS}\|_{\frac{4s}{4-s}} + \|D^2 \mathbf{v}_{NS}\|_s \le c , \end{aligned} \tag{A.6}$$

with $c = c(\Omega, s, \sigma)$.

Concerning problem (3.1) we have the following result.

Lemma A.5. *Let* $\varepsilon_0 > 0$. *There exist positive numbers* Re_0, *and* We_0, *such that for any* $0 < \mathrm{Re} \le \mathrm{Re}_0$, $0 < \mathrm{We} \le \mathrm{We}_0$, $0 < \varepsilon < \varepsilon_0$, *problem* (3.1) *has at least one solution. Moreover, there is a positive constant* $C = C(\Omega, t, q, \mathrm{We}_0, \mathrm{Re}_0, \varepsilon_0)$ *such that*

$$\begin{aligned} &\mathrm{Re}^{\frac{1}{2}} \|\mathbf{v} + \mathbf{U}\|_{\frac{2q}{2-q}} + \mathrm{Re}^{\frac{1}{4}} \|\operatorname{grad} \mathbf{v}\|_{\frac{4q}{4-q}} + \|D^2 \mathbf{v}\|_{1,q} + \|D^2 \mathbf{v}\|_{1,t} \\ &\qquad + \|\operatorname{grad} p\|_q + \|\operatorname{grad} p\|_t \le C , \end{aligned} \tag{A.7}$$

$q \in (1, 3/2)$, $t \in (1, \infty)$.[g] *Finally, for any given* $C_1 > 0$ *there exist* We_1, Re_1 *such that for all* $0 < \mathrm{Re} \le \mathrm{Re}_1$, $0 < \mathrm{We} \le \mathrm{We}_1$ *the solution is unique in the class of solutions satisfying* (A.7), *with* $C = C_1$.

[g]In the applications of Lemma A.5 given in this section, we will typically pick two different values for $t : t_1 < 3/2$, sufficiently close to 1, and $t_2 > 3/2$, sufficiently close to ∞.

Proof. The proof is relatively standard (see e.g. Ref. 31) and is based on the following idea (see Ref. 23). Consider the mapping

$$M : \mathbf{S} \mapsto (\mathbf{Z}, \pi) \mapsto \mathbf{s}\,,$$

where

$$\begin{aligned} &-\Delta \mathbf{z} + \mathrm{Re}\frac{\partial \mathbf{Z}}{x_1} + \operatorname{grad}\pi = \mathbf{S} \\ &\operatorname{div}\mathbf{Z} = 0 \end{aligned} \tag{A.8}$$

in Ω, with

$$\begin{aligned} &\mathbf{Z} = \mathbf{U} \quad at\ \Sigma \\ &\mathbf{Z} \to \mathbf{0} \quad |\mathbf{x}| \to \infty \end{aligned} \tag{A.9}$$

and

$$\begin{aligned} \mathbf{s} + \mathrm{We}\,(\mathbf{Z}-\mathbf{U})\cdot\operatorname{grad}\mathbf{s} = {}& -\mathrm{Re}\,\mathbf{Z}\cdot\operatorname{grad}\mathbf{Z} + \mathrm{We}\ \mathrm{Re}\frac{\partial^2\mathbf{Z}}{\partial x_1^2} \\ & + \mathrm{We}\ \operatorname{div}\Bigg[2\mathbf{D}(\mathbf{Z})\cdot(\operatorname{grad}\mathbf{Z})^{\mathrm{T}} + 2\varepsilon\mathbf{D}^2(\mathbf{Z}) \\ & + \mathrm{Re}\frac{\partial\mathbf{Z}}{\partial x_1}\otimes\mathbf{Z} - \pi(\operatorname{grad}\mathbf{Z})^{\mathrm{T}}\Bigg] \end{aligned} \tag{A.10}$$

in Ω, namely, $(\mathbf{Z}, \pi)$ solves the Oseen problem, and $\mathbf{s}$ solves the steady transport equation. Let $(\mathbf{w}, \phi)$ be the solution to (A.8)–(A.9) corresponding to the fixed point of the operator M. Then $(\mathbf{v}, p) = (\mathbf{w} - \mathbf{U}, \phi + \mathrm{We}(\mathbf{w} - \mathbf{U}) \cdot \operatorname{grad}\phi)$ solves (3.1). Now, combining estimates for the Oseen problem from Lemmas A.2 and A.3 we get the result using the contraction principle. Let us shortly sketch this procedure. Take $t_1 \in (1, 4/3]$ and $t_2 \in (3/2, \infty)$, sufficiently large, $q \in (1, 3/2)$. Take $K > 0$ and assume

$$\|\mathbf{S}\|_{1,q} + \|\mathbf{S}\|_{1,t_1} + \|\mathbf{S}\|_{1,t_2} \le K\,.$$

Set, further,

$$\begin{aligned} [|\mathbf{Z}, \pi|] = {}& \mathrm{Re}^{\frac{1}{2}}\|\mathbf{Z}\|_{\frac{2q}{2-q}} + \mathrm{Re}^{\frac{1}{4}}\|\operatorname{grad}\mathbf{Z}\|_{\frac{4q}{4-q}} + \mathrm{Re}^{\frac{1}{2}}\|\mathbf{Z}\|_{\frac{2t_1}{2-t_1}} + \mathrm{Re}^{\frac{1}{4}}\|\operatorname{grad}\mathbf{Z}\|_{\frac{4t_1}{4-t_1}} \\ & + \|D^2\mathbf{Z}\|_{1,q} + \|D^2\mathbf{Z}\|_{1,t_1} + \|D^2\mathbf{Z}\|_{1,t_2} \\ & + \|\operatorname{grad}\pi\|_{1,q} + \|\operatorname{grad}\pi\|_{1,t_1} + \|\operatorname{grad}\pi\|_{1,t_2}\,. \end{aligned}$$

Then, in view of (A.4),

$$[|\mathbf{Z}, \pi|] \le C(1 + K)\,.$$

Moreover, applying Lemma A.3 to (A.10) and observing that, by Sobolev embedding theorems, we have

$$\|\mathbf{Z}\|_{C^1(\bar\Omega)} \le c[|\mathbf{Z}, \pi|] \le cC(1 + K)\,,$$

with $c = c(\Omega, q, t_1, t_2)$, we can show that for We sufficiently small, the following estimate holds[h]

$$\|\mathbf{s}\|_{1,q} + \|\mathbf{s}\|_{1,t_1} + \|\mathbf{s}\|_{1,t_2} \leq C(\mathrm{Re}, \mathrm{We})(1 + [|\mathbf{Z}, \pi|]^2)\,,$$

where $C \to 0$ as $\mathrm{Re} \to 0$ and $\mathrm{We} \to 0$. We also easily prove that

$$\|\mathbf{s}_1 - \mathbf{s}_2\|_{t_1} + \|\mathbf{s}_1 - \mathbf{s}_2\|_q \leq C(\mathrm{Re}, \mathrm{We})(1 + K)\|\mathbf{S}_1 - \mathbf{S}_2\|_q$$

with $\mathbf{s}_i = M\mathbf{S}_i$, $i = 1, 2$, and the existence of unique solution to (3.1) in the ball $C(\mathrm{Re}, \mathrm{We})(1+[|\mathbf{Z}, \pi|]^2) \leq K$ is established. Note that we can take $[|\mathbf{Z}, \pi|]$ arbitrarily large; but then Re, We must be very small. □

Remark A.1. Problem (3.2) can be solved even without the smallness assumption on Re. Nevertheless, since we want to construct strong solution to (3.2), we can apply similar method and obtain an estimate corresponding to (A.7) (with the $W^{1,s}$-norm of $\mathrm{grad}\, p_{NS}$ instead of L^s-norm, and with C independent of We_0) in a very similar way as in Lemma A.5 (even easier).

We end our preliminary results by recalling the following useful interpolation result due to Maremonti (see Ref. 22 for more general version)

Lemma A.6. *Let* $\mathrm{grad}\,\mathbf{Z} \in L^s(\Omega)$, $s > 3$ *and* $\mathbf{Z} \in L^q(\Omega)$, $q \geq 1$. *Then there exist* $C = C(s, q, a)$ *and* $C_1 = C_1(\tau)$ *such that*

$$\|\mathbf{Z}\|_\infty \leq C(\|\mathrm{grad}\,\mathbf{Z}\|_s^a \|\mathbf{Z}\|_q^{1-a} + C_1\|\mathrm{grad}\,\mathbf{Z}\|_s^{a-\tau}\|\mathbf{Z}\|_q^{1-a+\tau})\,, \tag{A.11}$$

where $a \in [0, 1)$, $\tau \in (0, a]$ *and* $a(1/s - 1/3) + (1 - a)1/q = 0$.

We are now in a position to prove the main results.

Proof of Theorem 3.1. Setting $\mathbf{w} = \mathbf{v}_{NS} - \mathbf{v}_S$, from (3.2), (3.3) we find

$$\left.\begin{aligned} &\Delta\mathbf{w} = \mathrm{Re}\,\mathbf{v}_{NS}\cdot\mathrm{grad}\,\mathbf{v}_{NS} + \mathrm{grad}\,\Phi \\ &\mathrm{div}\,\mathbf{w} = 0 \end{aligned}\right\}\text{ in } \Omega$$
$$\mathbf{w} = 0 \quad \text{on } \Sigma \tag{A.12}$$
$$\lim_{|x|\to\infty} \mathbf{w}(x) = 0\,.$$

From Lemma A.1 and by the Hölder inequality we have, for all $1 < q < 3/2$,

$$\|\mathbf{w}\|_{\frac{3q}{3-2q}} + \|\mathrm{grad}\,\mathbf{w}\|_{\frac{3q}{3-q}} + \|D^2\mathbf{w}\|_q \leq c_3\,\mathrm{Re}\Big(\|\mathbf{v}_{NS} + \mathbf{U}\|_{\frac{2q}{2-q}}\|\mathrm{grad}\,\mathbf{v}_{NS}\|_2 + \|\mathrm{grad}\,\mathbf{v}_{NS}\|_q\Big)\,. \tag{A.13}$$

Choosing $q > 4/3$, we have $2q/(2 - q) \in (4, 6)$. Thus, applying Lemma A.4 (Eq. (A.6)), we obtain

$$\|\mathbf{v}_{NS} + \mathbf{U}\|_{\frac{2q}{2-q}}\|\mathrm{grad}\,\mathbf{v}_{NS}\|_2 \leq c_4\,. \tag{A.14}$$

[h]The bound $t_1 \leq 4/3$ comes from the estimate of the convective term.

Moreover, for any $4/3 < r \le 3/2$ we can find $\sigma > 3/2$, and $s \in (1, 12/11)$ such that $q_2 \equiv 4s/(4-s) < r$. Therefore, with this choice of q_1, by the convexity inequality and Lemma A.4, we get

$$\|\operatorname{grad}\mathbf{v}_{NS}\|_r \le \|\operatorname{grad}\mathbf{v}_{NS}\|_{q_2}^{\theta}\|\operatorname{grad}\mathbf{v}_{NS}\|_{\sigma}^{1-\theta} \le c_5\,\mathrm{Re}^{-\theta/4}\,, \quad \theta = \frac{q_2(r-\sigma)}{r(q_2-\sigma)}\,. \tag{A.15}$$

Setting $\eta = \theta/4$, from (A.15) and (A.14) with $r = q$, we deduce

$$\|\mathbf{w}\|_{\frac{3q}{3-2q}} + \|\operatorname{grad}\mathbf{w}\|_{\frac{3q}{3-q}} + \|D^2\mathbf{w}\|_q \le c\,\mathrm{Re}^{1-\eta}\,, \tag{A.16}$$

where η can be made close to zero, by fixing q_2 and by choosing q and σ close to $3/2$. Estimate (i) of Theorem 3.1 then follows from (A.16). To show estimate (ii), we observe that for any $\mathbf{u}_1, \mathbf{u}_2$, setting $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$, it is

$$\mathbf{S}(\mathbf{u}_1) - \mathbf{S}(\mathbf{u}_2) = 2(\mathbf{u}\cdot\operatorname{grad}\mathbf{D}(\mathbf{u}_1) + \mathbf{u}_2\cdot\operatorname{grad}\mathbf{D}(\mathbf{u})) + \mathbf{S}_1(\operatorname{grad}\mathbf{u}_1, \operatorname{grad}\mathbf{u}_2) \tag{A.17}$$

with

$$|\mathbf{S}_1(\operatorname{grad}\mathbf{u}_1, \operatorname{grad}\mathbf{u}_2)| \le C|\operatorname{grad}\mathbf{u}|(|\operatorname{grad}\mathbf{u}_1| + |\operatorname{grad}\mathbf{u}_2|) \tag{A.18}$$

and $C = C(\varepsilon_0)$. Using these formulas with $\mathbf{u}_1 = \mathbf{v}_{NS}$, $\mathbf{u}_2 = \mathbf{v}_S$ and applying Hölder's inequality we obtain

$$\begin{aligned}\|\mathbf{S}(\mathbf{v}_{NS}) - \mathbf{S}(\mathbf{v}_S)\|_q &\le \|\mathbf{w}\|_{\frac{3q}{3-2q}}\|\operatorname{grad}\mathbf{v}_{NS}\|_{\frac{3}{2}} + \|\mathbf{v}_S\|_\infty\|D^2\mathbf{w}\|_q \\ &\quad + C\|\operatorname{grad}\mathbf{w}\|_{\frac{3q}{3-q}}(\|\operatorname{grad}\mathbf{v}_{NS}\|_3 + \|\operatorname{grad}\mathbf{v}_S\|_3)\,.\end{aligned} \tag{A.19}$$

Clearly,

$$\|\operatorname{grad}\mathbf{v}_S\|_3 \le c_1\,, \tag{A.20}$$

and, from (A.15) with $r = 3/2$,

$$\|\operatorname{grad}\mathbf{v}_{NS}\|_{\frac{3}{2}} \le c_2\,\mathrm{Re}^{-\eta}\,. \tag{A.21}$$

Moreover, from Lemma A.4, we get

$$\|\operatorname{grad}\mathbf{v}_{NS}\|_3 \le c_3\,. \tag{A.22}$$

Thus, part (ii) of Theorem 3.1 follows from (A.19), (2.17) and (A.20)–(A.22). This concludes the proof of Theorem 3.1. □

Proof of Theorem 3.2. Estimate (i) is an easy consequence of (A.7) and (A.11). Namely

$$\|\mathbf{v}+\mathbf{U}\|_\infty \le C\left(\|\operatorname{grad}\mathbf{v}\|_{\frac{3t_2}{3-t_2}}^{a}\|\mathbf{v}+\mathbf{U}\|_{\frac{3q}{3-2q}}^{1-a} + C(\tau)\|\operatorname{grad}\mathbf{v}\|_{\frac{3t_2}{3-t_2}}^{a-\tau}\|\mathbf{v}+\mathbf{U}\|_{\frac{3q}{3-2q}}^{1-a+\tau}\right)$$

for some $t_2 > 3/2$ and $q < 3/2$. Then, since $\mathbf{v}+\mathbf{U} \in L^{\frac{2q}{2-q}}$ and $\operatorname{grad}\mathbf{v} \in L^{\frac{4q}{4-q}}$, we have

$$\|\mathbf{v}+\mathbf{U}\|_\infty \le C\left(\|D^2\mathbf{v}\|_t^a\|D^2\mathbf{v}\|_q^{1-a} + C(\tau)\|D^2\mathbf{v}\|_t^{a-\tau}\|D^2\mathbf{v}\|_s^{1-a+\tau}\right) \le C\,.$$

Further, to show (ii) and (iii), let us write the equation for the difference $\mathbf{v} - \mathbf{v}_{NS} \equiv \mathbf{u}$. We have

$$-\Delta\mathbf{u} + \mathrm{Re}\frac{\partial\mathbf{u}}{x_1} + \mathrm{grad}(p - p_{NS}) = -\mathrm{Rey}(\mathbf{v} + \mathbf{U}) \cdot \mathrm{grad}\,\mathbf{u} - \mathrm{Re}\,\mathbf{u} \cdot \mathrm{grad}\,\mathbf{v}_{NS}$$

$$+ 2\mathrm{We}\ \mathrm{grad}(\mathbf{A}_1(\mathbf{v}) + 2\varepsilon\mathbf{D}^2(\mathbf{v})) \equiv \mathbf{F}$$

$$\mathrm{div}\,\mathbf{v} = 0\,,$$

and

$$\mathbf{u} = \mathbf{0} \quad \text{at } \Sigma$$

$$\lim_{|\mathbf{x}|\to\infty} \mathbf{u}(x) = \mathbf{0}\,.$$

Applying estimate (A.4) with $\delta = 0$ we have

$$\mathrm{Re}^{\frac{1}{2}}\|\mathbf{u}\|_{\frac{2q}{2-q}} + \mathrm{Re}^{\frac{1}{4}}\|\mathrm{grad}\,\mathbf{u}\|_{\frac{4q}{4-q}} + \|D^2\mathbf{u}\|_q \le C\|\mathbf{F}\|_q\,. \tag{A.23}$$

Now, using (A.7) for both $\mathbf{v}$ and $\mathbf{v}_{NS}$, we have

$$\begin{aligned}\mathrm{Re}\|(\mathbf{v} + \mathbf{U}) \cdot \mathrm{grad}\,\mathbf{u}\|_q &\le \mathrm{Re}^{\frac{1}{4}}\|\mathrm{grad}\,\mathbf{u}\|_{\frac{4q}{4-q}}\mathrm{Re}^{\frac{3}{4}}\|\mathbf{v} + \mathbf{U}\|_4 \\ &\le \mathrm{Re}^{\frac{3}{4}}\|D^2\mathbf{v}\|_{\frac{12}{11}}\mathrm{Re}^{\frac{1}{4}}\|\mathrm{grad}\,\mathbf{u}\|_{\frac{4q}{4-q}} \\ &\le \frac{1}{2C}\mathrm{Re}^{\frac{1}{4}}\|\mathrm{grad}\,\mathbf{u}\|_{\frac{4q}{4-q}}\end{aligned}$$

for an appropriately chosen t_1, and for Re sufficiently small. Analogously,

$$\begin{aligned}\mathrm{Re}\|\mathbf{u} \cdot \mathrm{grad}\,\mathbf{v}_{NS}\|_q &\le \mathrm{Re}^{\frac{1}{2}}\|\mathbf{u}\|_{\frac{2q}{2-q}}\mathrm{Re}^{\frac{1}{2}}\|\mathrm{grad}\,\mathbf{v}_{NS}\|_2 \\ &\le \mathrm{Re}^{\frac{1}{2}}\|\mathbf{u}\|_{\frac{2q}{2-q}}\mathrm{Re}^{\frac{1}{2}}\|D^2\mathbf{v}_{NS}\|_{\frac{6}{5}} \\ &\le \frac{1}{2C}\mathrm{Re}^{\frac{1}{2}}\|\mathbf{u}\|_{\frac{2q}{2-q}}\,.\end{aligned}$$

The other two terms are now easily estimated by

$$C\,\mathrm{We}(\|D^2\mathbf{v}\|_{1,t_1}^2 + \|D^2\mathbf{v}\|_{1,t_2}^2 + 1)$$

for some t_1, t_2 appropriately chosen. Thus, from (A.23) we have

$$\mathrm{Re}^{\frac{1}{2}}\|\mathbf{u}\|_{\frac{2q}{2-q}} + \mathrm{Re}^{\frac{1}{4}}\|\mathrm{grad}\,\mathbf{u}\|_{\frac{4q}{4-q}} + \|D^2\mathbf{u}\|_q \le C\,\mathrm{We}\,.$$

Since $1 < q < 3/2$, we find

$$\|\mathbf{u}\|_{\frac{3q}{3-2q}} \le C\|D^2\mathbf{u}\|_q \le C\,\mathrm{We}$$

and (ii) is proved. To show (iii), we use (A.17) and (A.18) with $\mathbf{u}_1 \equiv \mathbf{v}$ and $\mathbf{u}_2 \equiv \mathbf{v}_{NS}$. We get

$$\mathbf{S}(\mathbf{v}) - \mathbf{S}(\mathbf{v}_{NS}) = 2(\mathbf{u} \cdot \mathrm{grad}\,\mathbf{D}(\mathbf{v}) + \mathbf{v}_{NS} \cdot \mathrm{grad}\,\mathbf{D}(\mathbf{u})) + \mathbf{S}_1(\mathrm{grad}\,\mathbf{v}, \mathrm{grad}\,\mathbf{v}_{NS})\,,$$

where

$$|\mathbf{S}_1(\operatorname{grad}\mathbf{v}, \operatorname{grad}\mathbf{v}_{NS})| \le C|\operatorname{grad}\mathbf{u}|(|\operatorname{grad}\mathbf{v}| + |\operatorname{grad}\mathbf{v}_{NS}|)\,.$$

Now[i]

$$\|\mathbf{v}_{NS}\cdot\operatorname{grad}\mathbf{D}(\mathbf{u})\|_q \le C\|D^2\mathbf{u}\|_q\|\mathbf{v}_{NS}\|_\infty \le C(1+\|\mathbf{v}_{NS}+\mathbf{U}\|_\infty)\|D^2\mathbf{u}\|_q \le C\,\mathrm{We}\,,$$

$$\|\,|\operatorname{grad}\mathbf{u}|\,|\operatorname{grad}\mathbf{v}_{NS}|\|_q \le \|\operatorname{grad}\mathbf{u}\|_{\frac{3q}{3-q}}\|\operatorname{grad}\mathbf{v}_{NS}\|_3 \le C\|D^2\mathbf{u}\|_q \le C\,\mathrm{We}\,.$$

Similarly,

$$\|\mathbf{u}\cdot\operatorname{grad}\mathbf{D}(\mathbf{v})\|_q \le \|\mathbf{u}\|_{\frac{3q}{3-2q}}\|D^2\mathbf{v}\|_{\frac{3}{2}} \le C\|D^2\mathbf{u}\|_q\|D^2\mathbf{v}\|_{\frac{3}{2}} \le C\,\mathrm{We}$$

$$\|\,|\operatorname{grad}\mathbf{u}|\,|\operatorname{grad}\mathbf{v}|\|_q \le \|\operatorname{grad}\mathbf{u}\|_{\frac{3q}{3-q}}\|\operatorname{grad}]\mathbf{v}\|_3 \le C\|D^2\mathbf{u}\|_q\|D^2\mathbf{v}\|_{\frac{3}{2}} \le C\,\mathrm{We}$$

and also (iii) is shown. The theorem is proved. □

Appendix B. Proof of Theorem 3.1

We assume $a \equiv x_1$. Denote by $\mathcal{C}_C^{(s)}$, some $C > 0$, the subclass of $\mathcal{C}_C$ consituted by vectors $\mathbf{v}$ and scalars p such that (see (4.5))

$$\mathbf{v} \in \mathcal{C}_1\,, \quad p = \mathcal{P}_2 p = \mathcal{P}_3 p\,.$$

By a direct inspection, we find that, in the class $\mathcal{C}_C^{(s)}$, it is

$$\mathbf{U} = U\mathbf{e}_1\,,$$

$$\int_\Sigma \mathbf{x}\times\mathbf{T}(\mathbf{v},p)\cdot\mathbf{n} = 0\,,$$

$$\int_\Sigma \mathbf{T}(\mathbf{v},p)\cdot\mathbf{n} = \eta\mathbf{e}_1\,, \quad \text{for some } \eta\in\mathbb{R}\,.$$

Moreover, dot-multiplying both sides of $(4.1)_1$ by $\mathbf{h}^{(1)} + \mathbf{e}_1$ and integrating by parts over Ω we get

$$\mathbf{e}_1\cdot\int_\Sigma \mathbf{T}(\mathbf{v},p)\cdot\mathbf{n} = \mathrm{We}\int_\Omega \mathbf{S}(\mathbf{v}):\mathbf{D}(\mathbf{h}) + \int_\Omega \mathbf{D}(\mathbf{v}):\mathbf{D}(\mathbf{h})$$
$$+\,\mathrm{Re}\int_\Omega \mathbf{v}\cdot\operatorname{grad}\mathbf{v}\cdot(\mathbf{h}+\mathbf{e}_1) + |\mathcal{B}|\mathbf{g}\cdot\mathbf{e}_1\,,$$

where, for simplicity, the subscript "(1)" has been omitted, and where $|\mathcal{B}|$ is the volume of the body $\mathcal{B}$. Moreover, dot-multiplying both sides of (4.3) (with $i = 1$) by $\mathbf{v}+\mathbf{U}$ and integrating by parts over Ω we find

$$\int_\Omega \mathbf{D}(\mathbf{v}):\mathbf{D}(\mathbf{h}) = \mathbf{U}\cdot\int_\Sigma \mathbf{T}(\mathbf{h},p^{(1)})\cdot\mathbf{n}\,.$$

[i]Note that for $\|\mathbf{v}_{NS}+\mathbf{U}\|_\infty$ we can get the same estimate as for $\|\mathbf{v}+\mathbf{U}\|_\infty$.

Therefore, given $\mathbf{g} = g\mathbf{e}_1$, $g > 0$ (say) the steady fall problem (4.1) in the class $\mathcal{C}_C^{(s)}$ is equivalent to find a triple $\{\mathbf{v}, P, U\}$ such that

$$\left.\begin{array}{l}\mathrm{We}\,\mathrm{div}\,\mathbf{S}(\mathbf{v}) + \mathrm{div}\,\mathbf{T}_N(\mathbf{v}, P) = \mathrm{Re}\mathbf{v}\cdot\mathrm{grad}\,\mathbf{v}\\ \mathrm{div}\,\mathbf{v} = 0\end{array}\right\}\text{ in }\Omega$$
$$\mathbf{v} = \mathbf{0}\quad\text{on }\Sigma$$
$$\lim_{|x|\to\infty}\mathbf{v}(x) = -U\mathbf{e}_1 \tag{B.1}$$
$$KU = m_e g - \mathrm{Re}\int_\Omega \mathbf{v}\cdot\mathrm{grad}\,\mathbf{v}\cdot(\mathbf{h}+\mathbf{e}_1) - \mathrm{We}\int_\Omega \mathbf{S}(\mathbf{v}) : \mathbf{D}(\mathbf{h})\,,$$

where

$$K = \mathbf{e}_1\cdot\int_\Sigma \mathbf{T}_N(\mathbf{h}, p^{(1)})\cdot\mathbf{n}\,,$$

$P = p - gx_1$, and $m_e = m - |\mathcal{B}|$ is the "effective mass", which we are assuming positive (for sedimentation to occur). As is well known[14] the number K is strictly positive. A solution to (B.1) can be obtained as a fixed point of a suitable map. Actually, let us consider the following equations:

$$\left.\begin{array}{l}\mathrm{We}\,\tilde{U}\,\mathrm{div}\,\mathbf{S}(\mathbf{u}) + \mathrm{div}\,\mathbf{T}_N(\mathbf{u}, \pi) = \mathrm{Re}\,\tilde{U}\mathbf{u}\cdot\mathrm{grad}\,\mathbf{u}\\ \mathrm{div}\,\mathbf{u} = 0\end{array}\right\}\text{ in }\Omega$$
$$\mathbf{u} = \mathbf{0}\quad\text{on }\Sigma \tag{B.2}$$
$$\lim_{|x|\to\infty}\mathbf{u}(x) = -\mathbf{e}_1$$

and

$$KU = m_e g - \mathrm{Re}\,\tilde{U}^2\int_\Omega \mathbf{u}\cdot\mathrm{grad}\,\mathbf{u}\cdot(\mathbf{h}+\mathbf{e}_1) - \mathrm{We}\,\tilde{U}^2\int_\Omega \mathbf{S}(\mathbf{u}) : \mathbf{D}(\mathbf{h})\,. \tag{B.3}$$

Let $T : \tilde{U} \mapsto U$ be the composition of the maps T_1 and T_2 defined as follows: for a given $\tilde{U} \in (U_0/8, U_0)$, some fixed $U_0 > 0$, T_1 maps $\tilde{U}$ in the solution $(\mathbf{u}, \pi)$ to (B.2), while T_2 maps $(\mathbf{u}, \pi)$ into U defined through (B.3). If we show that T possesses a fixed point, i.e. $\tilde{U} = U$, then a solution to the problem (B.1) is obtained by setting $\mathbf{v} = U\mathbf{u}$, $P = U\pi$. The choice of 1/8 as coefficient of U_0 is made for the sake of definiteness. We might have chosen as interval of definition of the map T any interval of the type (aU_0, bU_0), with $0 < a < b$.

For $q \in (1, 3/2)$, $t \in (1, \infty)$, set

$$\begin{aligned}\|\mathbf{u}, \pi\|_{q,t} = {} & (\mathrm{Re}\,\tilde{U})^{\frac{1}{2}}\|\mathbf{u}+\mathbf{e}_1\|_{\frac{2q}{2-q}} + (\mathrm{Re}\,\tilde{U})^{\frac{1}{4}}\|\mathrm{grad}\,\mathbf{u}\|_{\frac{4q}{4-q}}\\ & + \|D^2\mathbf{u}\|_{1,q} + \|D^2\mathbf{u}\|_{1,t} + \|\mathrm{grad}\,\pi\|_q + \|\mathrm{grad}\,\pi\|_t\,,\end{aligned}$$

and fix $\varepsilon > 0$. From Lemma A.5 we have that there exist positive constants γ_1, γ_2 (depending on q, t, ε) such that (B.2) has one and only one solution satisfying

$$\|\mathbf{u}, \pi\|_{q,t} < \gamma_1\,, \tag{B.4}$$

whenever $(\mathrm{Re}+\mathrm{We})U_0 < \gamma_2$. Reasoning exactly as in the proof of parts (i) and (ii) of Theorem 3.2, we show that

$$\|\mathbf{u}\|_\infty + \|\mathbf{S}(\mathbf{u})\|_q \le \gamma\,,$$

where γ is a positive constant independent of Re, We and $\tilde{U}$. Moreover, we recall the well-known summability properties for the Stokes velocity field[10] $\mathbf{h}$:

$$\begin{aligned} &(\mathbf{h}+\mathbf{e}_1) \in L^s(\Omega)\,, && \text{for all } s>3\\ &\mathbf{D}(\mathbf{h}) \in L^\sigma(\Omega)\,, && \text{for all } \sigma > 3/2\,. \end{aligned}$$

Therefore, from this latter inequality, from (B.3) and from the Hölder inequality we find

$$\begin{aligned} |KU - m_e g| &\le \mathrm{Re}\,\tilde{U}^2\|\mathbf{u}\|_\infty \|\mathrm{grad}\,\mathbf{u}\|_{s'}\|\mathbf{h}+\mathbf{e}_1\|_s + \mathrm{We}\,\tilde{U}^2\|\mathbf{S}(\mathbf{u})\|_{\sigma'}\|\mathbf{D}(\mathbf{h})\|_\sigma\\ &\le \kappa(\mathrm{Re}\,\tilde{U}^2\|\mathrm{grad}\,\mathbf{u}\|_{\sigma'} + \mathrm{We}\,\tilde{U}^2)\,, \end{aligned} \tag{B.5}$$

where κ is independent of Re, We and $\tilde{U}$. Since σ' is arbitrary in $(1, 3)$, Eqs. (B.4) and (B.5) imply

$$|KU - m_e g| \le \kappa_1(\mathrm{Re}^{\frac{3}{4}}\tilde{U}^{\frac{7}{4}} + \mathrm{We}\tilde{U}^2)\,. \tag{B.6}$$

Choose $U_0 = 2m_e g/K$. Then, for $\tilde{U} \in (0, U_0)$, and for Re and We sufficiently small (B.6) furnishes

$$|U - U_0/2| \le \frac{\kappa_1}{K}((\mathrm{Re}^{\frac{3}{4}}U_0^{\frac{7}{4}} + \mathrm{We}U_0^2) \le U_0/4\,.$$

From this relation we find

$$\frac{1}{4}U_0 \le U \le \frac{3}{4}U_0\,, \tag{B.7}$$

which implies, in particular, that the map T trasforms the interval $(U_0/8, U_0)$ in itself. It remains to prove that T is a contraction. To this end, we define $\mathbf{u}'$ and ϕ as follows:

$$\mathbf{u} = \mathbf{u}' - \mathbf{e}_1\,, \quad \pi = \phi + \mathrm{We}\,\tilde{U}\mathbf{u}\cdot\mathrm{grad}\,\phi\,.$$

Since $\mathrm{grad}\,\pi \in L^t(\omega)$ for all $t \in (1,\infty)$, in view of Lemma A.3 the field ϕ exists in a suitable functional class if $\mathrm{We}\,\tilde{U}$ is sufficiently small. Consequently, (B.2) can be equivalently written as follows (see (A.8)–(A.10):

$$\begin{gathered} -\Delta\mathbf{u}' + \mathrm{Re}\,\tilde{U}\frac{\partial\mathbf{u}'}{x_1} + \mathrm{grad}\,\phi = \mathbf{w}\,,\\ \mathrm{div}\,\mathbf{u}' = 0\,,\\ \mathbf{u}' = \mathbf{e}_1 \quad \text{at } \Sigma\,, \quad \mathbf{u}' \to \mathbf{0} \quad \text{as } |x| \to \infty\,. \end{gathered} \tag{B.8}$$

$$\mathbf{w} + \mathrm{We}\,\tilde{U}(\mathbf{u}' - \mathbf{e}_1)\cdot \mathrm{grad}\,\mathbf{w} = -\mathrm{Re}\,\tilde{U}\mathbf{u}'\cdot \mathrm{grad}\,\mathbf{u}' + \mathrm{We}\ \mathrm{Re}\,\tilde{U}^2 \frac{\partial^2 \mathbf{u}'}{\partial x_1^2}$$
$$+ \mathrm{We}\,\tilde{U}\,\mathrm{div}\Bigg[2\mathbf{D}(\mathbf{u}')\cdot(\mathrm{grad}\,\mathbf{u}')^{\mathrm{T}} + 2\varepsilon\mathbf{D}^2(\mathbf{u}')$$
$$+ \mathrm{Re}\,\tilde{U}\frac{\partial \mathbf{u}'}{x_1}\otimes \mathbf{u}' - \phi(\mathrm{grad}\,\mathbf{u}')^{\mathrm{T}}\Bigg].$$

Let $(\mathbf{u}_1, \phi_1)$ and $(\mathbf{u}_2, \phi_2)$ be the two solutions to (B.8) corresponding to $\tilde{U}_1$ and $\tilde{U}_2$, respectively. Setting

$$\mathbf{V} = \mathbf{u}_1 - \mathbf{u}_2\,, \quad \tilde{U} = \tilde{U}_1 - \tilde{U}_2\,, \quad \phi = \phi_1 - \phi_2\,, \quad \mathbf{w} = \mathbf{w}_1 - \mathbf{w}_2$$

we find

$$\begin{gathered} -\Delta\mathbf{V} + \mathrm{Re}\,\tilde{U}_1\frac{\partial \mathbf{V}}{x_1} + \mathrm{grad}\,\phi = \mathbf{w} - \mathrm{Re}\,\tilde{U}\frac{\partial \mathbf{u}_2}{x_1}\,, \\ \mathrm{div}\,\mathbf{V} = 0\,, \\ \mathbf{V} = \mathbf{0} \quad \text{at } \Sigma\,, \quad \mathbf{V}(x) \to \mathbf{0} \quad \text{as } |x| \to \infty\,. \end{gathered} \tag{B.9}$$

$$\mathbf{w} + \mathrm{We}\,\tilde{U}_1(\mathbf{u}_1 - \mathbf{e}_1)\cdot \mathrm{grad}\,\mathbf{w} = \mathrm{We}\,\mathbf{F}_1 + \mathrm{Re}\,\mathbf{F}_2 + \mathrm{We}\ \mathrm{Re}\,\mathbf{F}_3 + \mathrm{We}\ \mathrm{div}\,\mathbf{F} \equiv \mathbf{f}\,,$$

where the vector functions $\mathbf{F}_i$, $i = 1, 2, 3$ are given by

$$\begin{aligned} \mathbf{F}_1 &= \tilde{U}(\mathbf{u}_1 - \mathbf{e}_1)\cdot \mathrm{grad}\,\mathbf{w}_2 + \tilde{U}_2\mathbf{V}\cdot \mathrm{grad}\,\mathbf{w}_2 \\ \mathbf{F}_2 &= \tilde{U}_1\mathbf{V}\cdot \mathrm{grad}\,\mathbf{u}_1 + \tilde{U}_1\mathbf{u}_2\cdot \mathrm{grad}\,\mathbf{V} + \tilde{U}\mathbf{u}_2\cdot \mathrm{grad}\,\mathbf{u}_2 \\ \mathbf{F}_3 &= \tilde{U}_1^2\frac{\partial^2 \mathbf{V}}{\partial x_1^2} + (\tilde{U}_1^2 - \tilde{U}_2^2)\frac{\partial^2 \mathbf{u}_1}{\partial x_1^2}\,, \end{aligned}$$

while the tensor function $\mathbf{F}$ is defined as follows:

$$\begin{aligned} \mathbf{F} = {} & 2(\tilde{U}_1\mathbf{D}(\mathbf{V})\cdot(\mathrm{grad}\,\mathbf{u}_1)^{\mathrm{T}} + \tilde{U}_1\mathbf{D}(\mathbf{u}_2)\cdot(\mathrm{grad}\,\mathbf{V})^{\mathrm{T}} + \tilde{U}\mathbf{D}(\mathbf{u}_2)\cdot(\mathrm{grad}\,\mathbf{u}_2)^{\mathrm{T}}) \\ & + 2\varepsilon(\tilde{U}_1\mathbf{D}(\mathbf{V})\cdot\mathbf{D}(\mathbf{u}_1) + \tilde{U}_1\mathbf{D}(\mathbf{u}_2)\cdot\mathbf{D}(\mathbf{V}) + \tilde{U}\mathbf{D}^2(\mathbf{u}_2)) \\ & + \mathrm{Re}\left(\tilde{U}_1^2\frac{\partial \mathbf{V}}{\partial x_1}\otimes \mathbf{u}_1 + \tilde{U}_1^2\frac{\partial \mathbf{u}_2}{\partial x_1}\otimes \mathbf{V} + (\tilde{U}_1^2 - \tilde{U}_2^2)\frac{\partial \mathbf{u}_2}{\partial x_1}\otimes \mathbf{u}_2\right) \\ & - \tilde{U}_1\phi(\mathrm{grad}\,\mathbf{u}_1)^{\mathrm{T}} - \tilde{U}_1\phi_2(\mathrm{grad}\,\mathbf{V})^{\mathrm{T}} - \tilde{U}\phi_2(\mathrm{grad}\,\mathbf{u}_2)^{\mathrm{T}}\,. \end{aligned}$$

From Lemma A.3 applied to $(\mathrm{B.9})_4$, and using (B.4), for We sufficiently small we find

$$\|\mathbf{w}\|_q \le C\|\mathbf{f}\|_q\,. \tag{B.10}$$

We wish to estimate the right-hand side of (B.10). Set

$$\langle\langle \mathbf{V}, \phi\rangle\rangle = (\mathrm{Re}\,\tilde{U}_1)^{\frac{1}{2}}\|\mathbf{V}\|_{\frac{2q}{2-q}} + (\mathrm{Re}\,\tilde{U}_1)^{\frac{1}{4}}\|\mathrm{grad}\,\mathbf{V}\|_{\frac{4q}{4-q}} + \|D^2\mathbf{V}\|_{1,q} + \|\mathrm{grad}\,\phi\|_q\,.$$

Taking into account (B.4), that $\tilde{U}_i \in (U_0/8, U_0)$, $i = 1, 2$, and using the following Sobolev-type estimates[10]

$$\|\mathbf{V}\|_{\frac{3q}{3-2q}} + \|\mathrm{grad}\,\mathbf{V}\|_{\frac{3q}{3-q}} \le \gamma \|D^2\mathbf{V}\|_q\,, \quad 1 < q < \frac{3}{2} \tag{B.11}$$

by a straightforward calculation that also uses several times the Hölder inequality we can prove the validity of the following relation for all $q \in (1, 3/2)$

$$\|\mathbf{f}\|_q \le C_0(\mathrm{Re}, \mathrm{We})(\langle\langle \mathbf{V}, \phi \rangle\rangle + |\tilde{U}|)\,, \tag{B.12}$$

where $C_0(\mathrm{Re}, \mathrm{We}) \to 0$ as $\mathrm{Re}, \mathrm{We} \to 0$. From Lemma A.2 applied to $(\mathrm{B}.9)_{1,2,3}$ we also deduce

$$\langle\langle \mathbf{V}, \phi \rangle\rangle \le C(\|\mathbf{w}\|_q + \mathrm{Re}|\tilde{U}|\,\|\mathrm{grad}\,\mathbf{u}_2\|_q)\,. \tag{B.13}$$

As a consequence, choosing $q \in (4/3, 3/2)$, from (B.4), (B.10), (B.12) and (B.13) it follows, for sufficiently small Re and We, that

$$\langle\langle \mathbf{V}, \phi \rangle\rangle \le \gamma_0 |\tilde{U}|\,, \tag{B.14}$$

for a suitable positive constant γ_0 which can be made independent of Re, We and $\tilde{U}$. Next, from (B.3) we have that $U \equiv T(\tilde{U}_1) - T(\tilde{U}_2)$ satisfies the following equation:

$$\begin{aligned} KU = &-\mathrm{Re} \int_\Omega \{\tilde{U}_1^2(\mathbf{V} \cdot \mathrm{grad}\,\mathbf{u}_1 + (\mathbf{u}_1 + \mathbf{e}_1) \cdot \mathrm{grad}\,\mathbf{V}) \\ &+ (\tilde{U}_1^2 - \tilde{U}_2^2)(\mathbf{u}_2 \cdot \mathrm{grad}\,\mathbf{u}_2)\} \cdot (\mathbf{h} + \mathbf{e}_1) \\ &- \mathrm{We} \int_\Omega \{\tilde{U}_1^2(\mathbf{S}(\mathbf{u}_1 + \mathbf{e}_1) - \mathbf{S}(\mathbf{u}_2 + \mathbf{e}_1)) - (\tilde{U}_1^2 - \tilde{U}_2^2)\mathbf{S}(\mathbf{u}_2 + \mathbf{e}_1)\} : \mathbf{D}(\mathbf{h})\,. \end{aligned} \tag{B.15}$$

Using (B.11) and (B.4) we easily establish, for that

$$\begin{aligned} &\|\tilde{U}_1^2(\mathbf{V} \cdot \mathrm{grad}\,\mathbf{u}_1 + (\mathbf{u}_1 + \mathbf{e}_1) \cdot \mathrm{grad}\,\mathbf{V}) + (\tilde{U}_1^2 - \tilde{U}_2^2)(\mathbf{u}_2 \cdot \mathrm{grad}\,\mathbf{u}_2)\|_q \\ &\quad \le c_1\,\mathrm{Re}^{-\frac{1}{2}}(\langle\langle \mathbf{V}, \phi \rangle\rangle + |\tilde{U}|)\,. \end{aligned} \tag{B.16}$$

Furthermore, again from (B.4) and (A.17), (A.18) we get

$$\begin{aligned} &\|(\mathbf{S}(\mathbf{u}_1 + \mathbf{e}_1) - \mathbf{S}(\mathbf{u}_2 + \mathbf{e}_1)) - (\tilde{U}_1^2 - \tilde{U}_2^2)\mathbf{S}(\mathbf{u}_2 + \mathbf{e}_1)\|_q \\ &\quad \le c_2(\langle\langle \mathbf{V}, \phi \rangle\rangle + |\tilde{U}|)\,. \end{aligned} \tag{B.17}$$

In Eqs. (B.16) and (B.17) the constants c_1, c_2 are independent of Re and We. Thus, from (B.15), with the help of (B.14), (B.16) and (B.17) we conclude that

$$|U| \le c_0(\mathrm{Re}, \mathrm{We})|\tilde{U}|\,,$$

where $c_0(\mathrm{Re}, \mathrm{We}) \to 0$ as $\mathrm{Re}, \mathrm{We} \to 0$. This shows that T is a contraction and, therefore, the existence part of Theorem 3.1 is accomplished. The uniqueness part is immediately obtained as follows. The fields $(\mathbf{v}, p)$ and $(\mathbf{v}_1, p_1)$ are solutions to (4.1) corresponding to the same $\mathbf{U}$ and in the same class $\mathcal{C}_C$. Therefore, by Lemma A.5, the two solutions coincide for Re and We sufficiently small, and therefore, from $(4.1)_5$, it follows $\mathbf{g} = \mathbf{g}_1$, and the proof of the theorem is completed. □

Acknowledgments

Part of the research presented in this paper was developed while G.P.G. was a Visiting Professor at the Department of Mathematics of the Second University of Naples, Caserta, Italy, in May 2000. He would like to thank Professors Remigio Russo, Paolo Maremonti and Giulio Starita for their kind invitation, warm hospitality and for the intellectually stimulating atmosphere they were able to establish. The research of G.P.G. and A.V. was partially supported by the NSF grant DMS-0103970. The research of M.P. was partially supported by the Grant Agency of the Czech Republic (grant No. 201/00/0768) and by the Council of the Czech Governement (project No. 113200007).

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