Irrotational analysis of the toroidal bubble in a viscous fluid

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Abstract

We consider the problem of the rise of a toroidal gas bubble previously considered by Pedley (1968). We add the irrotational viscous drag to the force wrench in the impulse equation. In this case, the impulse equation and the energy equation governing the rise of the bubble are the same. The solution of this equation is computed; after a transient state the system evolves to a steady state in which the diameter, toroidal radius and rise velocity are constant.

1 Introduction

Turner (1957) developed a theory for the motion of a buoyant ring in an inviscid liquid. The theory shows that the buoyant force acts to increase the impulse of a ring. The ring diameter increases as the ring rises. He also carried out experiments to verify the theory with small vortex rings formed in water, using methylated spirits and salt to produce the density differences.

Walters & Davidson (1963) observed that a rising toroidal bubble could be produced by release of a mass of gas in water. The form of the bubble is a vortex ring with a buoyant air core. The experiments of Walters & Davidson (1963) motived Pedley's analysis (1968) for the toroidal bubble.

We will use the notation introduced by Pedley and summarized in figure 1. The radius of the ring is a and the plane of this ring is horizontal. The center of this ring is instantaneously at the point O and the axis Oz is the upward vertical. The cross section of the air core is assumed to be a circle with radius b. The circulation around the air core is $2\pi\Gamma$. (s, χ) are polar coordinates in a meridional plane, centered at the point A. The bubble is rising vertically with velocity U and the radius a can be expanding at the same time. The volume of the bubble V is given by $2\pi^2 ab^2$.

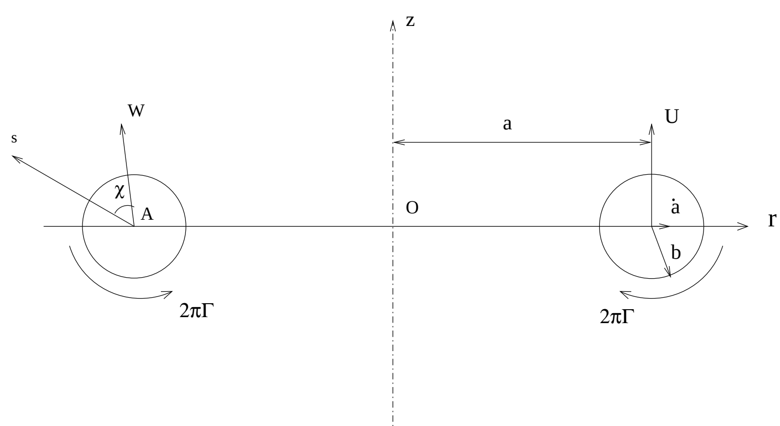

Figure 1: Meridional section of the toroidal bubble.

Pedley analyzed the motion of a toroidal bubble by two methods, the impulse equation and the energy equation. The rate of change of the vertical impulse is

$$\mathrm{d}P/\mathrm{d}t = F, \tag{1}$$

where F is the resultant force on the bubble in the vertical direction. Pedley argued that in this case, F is due to the buoyancy of the bubble and there is no viscous drag. The energy equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}(T+\Omega) = -\mathcal{D}, \tag{2}$$

where T and Ω are the kinetic and potential energies of the system and $\mathcal{D}$ is the dissipation. Pedley wrote the rate of change of potential energy as

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\rho g V h) = -\rho g V U, \tag{3}$$

where ρ is the density of the ambient fluid, and h is the depth of the bubble beneath a fixed reference level.

Pedley first considered the ambient fluid to be inviscid and the motion to be irrotational. Under these assumptions, the impulse and energy equations give rise to the same expressions for a and U as functions of t: a ultimately increases as $t^{1/2}$, and U decreases as $t^{-1/2}\ln t$. Pedley then considered viscous fluid and assumed that the flow remained approximately irrotational. He argued that the impulse equation would be the same as for the inviscid fluid; however, the energy equation would be changed due to the viscous dissipation. Thus the two methods yielded conflicting results, which led Pedley to conclude that the flow could not remain approximately irrotational. Pedley then investigated the diffusion of vorticity from the bubble surface and showed that the vorticity distribution became approximately Gaussian, with an effective radius b' which also increased as $t^{1/2}$. The solution for a as a function of t turned out to be the same as in inviscid case, while U decreased as $t^{-1/2}$. Other results produced by Pedley's analysis include an evaluation of the effect of a hydrostatic variation in bubble volume, and a prediction of the time when the bubble becomes unstable under the action of surface tension.

We consider the toroidal bubble under the assumption that the flow remains approximately irrotational and the ambient fluid is viscous. Our analysis is purely irrotational and it is different than the one given by Pedley because we include the viscous drag on the bubble in the impulse equation (1). This viscous drag can be computed using the dissipation method, as was done by Levich (1949) in his calculation of the drag on a spherical gas bubble in the irrotational flow of a viscous fluid. When the drag is added, the inconsistency between the impulse equation and energy equation disappears. Our equations predict that initially the bubble expands and rises in the fluid. The rise velocity U decreases as the ring radius a increases. Ultimately the viscous drag balances the buoyant force and a steady solution is achieved, for which a and U become constants.

In our calculation, we assume that the circulation Γ and the bubble volume V are constant, and the ratio $b/a << 1$. The same assumptions were adopted by Pedley (1968).

2 The energy equation

The flow around the toroidal bubble is approximated by the irrotational flow around a cylindrical bubble of radius b. The translational velocities of the bubble include the vertical velocity U and horizontal velocity $\dot{a}$ (figure 1). We denote the resultant velocity as

$$W = \sqrt{U^2 + \dot{a}^2}. \tag{4}$$

The rise velocity U may be calculated from the condition that there is no normal velocity across the bubble surface. If the core cross-section is taken to be circular and the quantity ω/r uniform inside, where ω is the vorticity, Lamb (1932, §163) gives

$$U = \frac{\Gamma}{2a}\left[\ln\frac{8a}{b} - n\right], \tag{5}$$

where $n = 1/4$. Hicks (1884) gives the value $n = 1/2$ for his hollow vortex-ring in which the core cross-section is not circular and ω/r is not uniform. The irrotational flow around a cylindrical bubble of radius b is given by the velocity potential

$$\phi = -W\frac{b^2}{s}\cos\chi + \Gamma\chi, \tag{6}$$

where the coordinates s and χ are shown in figure 1. The velocities can be obtained from the potential

$$u_s = \frac{\partial\phi}{\partial s}, \quad u_\chi = \frac{1}{s}\frac{\partial\phi}{\partial\chi}. \tag{7}$$

The kinetic energy for a 2D section is

$$T' = \frac{1}{2}\rho\int_b^{C_0 a}\int_0^{2\pi}\left(u_s^2 + u_\chi^2\right) s\,\mathrm{d}s\mathrm{d}\chi \tag{8}$$

$$= \rho\pi\left[\frac{1}{2}W^2b^2\left(1 - \frac{b^2}{C_0^2a^2}\right) + \Gamma^2\ln\frac{C_0 a}{b}\right], \tag{9}$$

where the integration is from b to C_0a rather than b to ∞ because of the logarithmic singularity. Multiply by $2\pi a$, the circumference of the bubble, and obtain the total kinetic energy

$$T = \rho\pi^2 aW^2b^2\left(1 - \frac{b^2}{C_0^2a^2}\right) + 2\rho\pi^2 a\Gamma^2\ln\frac{C_0 a}{b}. \tag{10}$$

The value of C_0 may be determined by comparing (10) to the kinetic energy given by Pedley (1968). Pedley used Lamb's (1932) formula for the kinetic energy of an arbitrary axisymmetric distribution of azimuthal vorticity and obtained

$$T = 2\pi^2\rho a\Gamma^2\left[\ln\frac{8a}{b} - 2 + O\left(\frac{b^2}{a^2}\ln\frac{8a}{b}\right)\right] \approx 2\pi^2\rho a\Gamma^2\ln\left(\frac{8a}{b}\mathrm{e}^{-2}\right). \tag{11}$$

The kinetic energy given by Pedley neglected the energy associated with the translation of the bubble. Comparing this with the energy associated with the circulation in (10), we obtain

$$C_0 = 8\mathrm{e}^{-2} \approx 1.0827. \tag{12}$$

Since $b << a$, the energy (10) is approximately

$$T = \rho\pi^2 a\left(W^2b^2 + 2\Gamma^2\ln\frac{C_0 a}{b}\right). \tag{13}$$

The dissipation of the potential flow (6) can be evaluated as

$$\mathcal{D}' = \int_0^{2\pi}\int_b^{\infty} 2\mu\mathbf{D}:\mathbf{D}\,s\,\mathrm{d}s\mathrm{d}\chi = 8\pi\mu W^2 + \frac{4\pi\mu\Gamma^2}{b^2}, \tag{14}$$

where $\mathbf{D}$ is the symmetric part of the rate of strain tensor. The same dissipation was obtained by Ackeret (1952). After multiplying $2\pi a$, we obtain the total dissipation

$$\mathcal{D} = 8\pi^2 \mu a \left(2W^2 + \Gamma^2/b^2\right). \tag{15}$$

The part of the dissipation associated with the circulation Γ in (15) is the same as that given by Pedley(1968). Pedley did not consider the dissipation associated with the translation W.

After inserting the kinetic energy (13), potential energy (3) and the dissipation (15) into the energy equation (2), we obtain

$$\frac{1}{2}V\frac{\mathrm{d}W^2}{\mathrm{d}t} + \left(2\pi^2\Gamma^2\dot{a} - \frac{gV\Gamma}{2a}\right)\left(\ln\frac{8a}{b} - \frac{1}{2}\right) = -16\pi^2\nu a\left(W^2 + \pi^2 a\Gamma^2/V\right), \tag{16}$$

where we have used $n = 1/2$ in the expression for U (5). Since b can be eliminated from $V = 2\pi^2 ab^2$, a is the only unknown in equation (16). If the fluid is inviscid and the kinetic energy associated with translation is neglected, (16) reduces to

$$2\pi^2\Gamma^2\dot{a} - \frac{gV\Gamma}{2a} = 0 \quad \Rightarrow \quad a^2 = a_0^2 + \frac{gV}{2\pi^2\Gamma}t, \tag{17}$$

where a_0 is the value of a at $t = 0$. This equation is the same as the formula given by Pedley (1968) for both inviscid and viscous fluids.

The translational velocity W has the following form

$$W^2 = \frac{\Gamma^2}{4a^2}\left(\ln\frac{8a}{b} - \frac{1}{2}\right)^2 + \dot{a}^2. \tag{18}$$

The derivative of W^2 with respect to time in (16) leads to a second order ordinary differential equation for a. We specify two initial conditions for a at $t = 0$. The first condition is

$$a = a_0 \quad \text{at} \quad t = 0. \tag{19}$$

The second condition is derived from (17)

$$\dot{a}(t=0) = \frac{gV}{4\pi^2\Gamma a_0}. \tag{20}$$

This condition is justified if the energy and dissipation associated with the translation of the bubble is much smaller than those associated with the circulation, and the viscous effects are small at $t = 0$. We will verify these conditions in section 4 when we insert the parameters taken from the experiments of Walters & Davidson into our equations and compute the numerical results.

If the part of the energy and dissipation associated with W is relatively small at all times, (16) becomes

$$\left(2\pi^2\Gamma\dot{a} - \frac{gV}{2a}\right)\left(\ln\frac{8a}{b} - \frac{1}{2}\right) = -16\pi^4\nu a^2\frac{\Gamma}{V}. \tag{21}$$

This is the same energy equations considered by Pedley (1968) under the assumption that the flow remains approximately irrotational in a viscous fluid. Equation (21) is a first order ordinary differential equation for a and the condition (19) will suffice. In section 4 we will shown that the solutions of (16) and (21) are nearly the same when using the parameters taken from the experiments of Walters & Davidson.

3 The impulse equation

Pedley used Lamb's formula for the impulse of an arbitrary axisymmetric distribution of azimuthal vorticity and obtained

$$P = 2\pi^2 \rho a^2 \Gamma \left[1 + O\left(\frac{b^2}{a^2} \ln \frac{8a}{b} \right) \right]. \tag{22}$$

Pedley included the buoyant force in the impulse equation and argued that the drag does not enter into this problem even when the fluid is viscous. We propose to include the viscous drag on the bubble in the impulse equation; then the inconsistency between the impulse equation and the energy equation disappears.

The viscous drag on a bubble may be computed using the dissipation method, which equates the power of the drag to the dissipation. Levich (1949) used the dissipation method to calculate the drag on a spherical gas bubble rising in the irrotational flow of a viscous fluid. Ackeret (1952) used the dissipation method to compute the drag on a rotating cylinder in a uniform stream. If the flow around the toroidal bubble is approximated by the irrotational flow around a cylindrical bubble with circulation, the velocity potential (6) is the same as that used by Ackeret (1952). The irrotational dissipation is given by (15). In section 4 we will show that the dissipation associated with the translation W is small and $\dot{a}$ is much smaller than U when using the parameters taken from the experiments of Walters & Davidson. Therefore the drag in the vertical direction from the dissipation method is approximately

$$D = \frac{\mathcal{D}}{U} = 8\pi^2 \mu a \frac{\Gamma^2}{b^2 U}. \tag{23}$$

With this viscous drag, the impulse equation (1) becomes

$$2\pi^2 \rho \Gamma \frac{\mathrm{d}a^2}{\mathrm{d}t} = \rho g V - 8\pi^2 \mu a \frac{\Gamma^2}{b^2 U}. \tag{24}$$

After using the expression for U (5), we obtain

$$\left(2\pi^2 \Gamma \dot{a} - \frac{gV}{2a} \right) \left(\ln \frac{8a}{b} - \frac{1}{2} \right) = -16\pi^4 \nu a^2 \frac{\Gamma}{V}, \tag{25}$$

which is the same as the energy equation (21). Thus the impulse equation with the viscous drag included is the same as the energy equation with the viscous dissipation.

4 Results

We solve the equations (16) and (21) using the physical constants

$$g = 980 \mathrm{cm\,s^{-2}}, \quad \nu = 0.011 \mathrm{cm^2 s^{-1}}, \tag{26}$$

and the parameters taken from the experiments of Walters & Davidson

$$\Gamma = 50 \mathrm{cm^2 s^{-1}}, \quad V = 21 \mathrm{cm^3}, \quad a_0 = 2.5 \mathrm{cm}, \quad b_0 = 0.65 \mathrm{cm}. \tag{27}$$

These parameters, except a_0 and b_0, are the same as those used by Pedley (1968); Pedley estimated $a_0 = 5$ cm. Our estimate of a_0 is based on figure 7 of Walters & Davidson (1963) which shows the photos of the toroidal bubble rising close to the free surface at the top of the tank. The caption in their figure 7 indicates that the ring radius is about 4 cm. Because the ring expands as it rises, the initial ring radius should be less than 4 cm. We estimate a_0 to be 2.5 cm and $b_0/a_0 \approx 0.26$.

We solve equation (16) with the initial conditions (19) and (20), and equation (21) with (19). To highlight the viscous effects, we also solve (16) with $\nu = 0$; the initial conditions for the inviscid equation are still (19) and (20). The solutions for a as a function of time are plotted in figure 2. The rise velocity U is then computed from (5) and plotted in figure 3. Integration of U gives rise to the height $h - h_0$ of rise, where h_0 is the initial position of the bubble. The plots for $h - h_0$ against a are given in figure 4.

The solution of the inviscid equation shows that a increases and U decreases with time. These solutions are similar to those obtained by Pedley (1968) using irrotational flow of a inviscid fluid. The plot for $h - h_0$ against a from the inviscid solution demonstrates that the bubble rises and expands conically, which is consistent with the solutions of Turner (1957) and Pedley (1968). Figure 2 shows that the two viscous equation (16) and (21) lead to almost the same solution. This demonstrates that the energy and dissipation associated with the translation W is relatively small, and that the choice of the initial condition (20) does not have substantial effects on the solution. Figure 5 shows the ratio between the ring expansion velocity $\dot{a}$ and the rise velocity U obtained from the solution of (16). This plot demonstrates that $\dot{a}$ is much smaller than U. Initially the viscous solution is similar to the inviscid solution; $\dot{a}$ increases and U decreases with time. As a/b^2 increases, the viscous drag (23) increases and finally balances the buoyant force. Then a steady state is reached, in which the toroidal bubble keeps its shape and rises at a constant velocity. Our viscous solutions give $a = 10.9$ cm and $U = 11.8$ cm/s at the steady state. This steady state solution is consistent with the classical description of the motion of a vortex ring. For example, Milne-Thomson (1968) says in § 19.41

"We may, however, observe that for points in the plane of the ring (considered as of infinitesimal cross-section) there is no radial velocity. This follows at once from the Biot and Savart principle, explained in 19.23. It therefore follows that the radius of the ring remains constant, and the ring moves forward with a velocity which must be constant since the motion must be steady relatively to the ring."

The comparison of our solutions to the experiments of Walters & Davidson (1963) is inconclusive. Walters & Davidson did not provide data for the ring radius or the rise velocity as a function of time. The tank used in their experiments is only 3 ft (91.44 cm) tall. Their figure 8 shows the computed circulation vs. time for only 0.6 second. For such a short time, we cannot even tell the difference between the viscous and inviscid solutions in figures 2 and 3. It is not known whether the bubble will ultimately reach a steady state as predicted by our viscous solution, or the ring radius will keep growing until instability occurs as in Pedley's solution.

5 Conclusion

We carry out a purely irrotational analysis for the motion of a toroidal bubble in a viscous fluid assuming that the flow is approximately irrotational. The evolution of the energy equation is formed from epressions for energy and dissipation evaluated on potential flow. An impulse equation is also formed in which both the buoyant force and viscous drag are included in the force wrench on the bubble. The part of kinetic energy and dissipation associated with the translation of the bubble is small compared to the part associated with the circulation. If the part associated with the translation is neglected, the impulse and energy equations are the same and give rise to a solution, in which the bubble radius a increases initially as the bubble rises, but ultimately the bubble reaches a steady state and rises at a constant velocity with a constant radius. The viscous drag balances the buoyant force in the steady state. Our analysis leads to bubble dynamics for the toroidal bubble like other bubbles,such as a spherical bubble (Levich 1949, Moore 1963) or a spherical cap bubble (Davies & Taylor 1950, Joseph 2003), which ultimately rise with a constant velocity and shape in a viscous fluid.

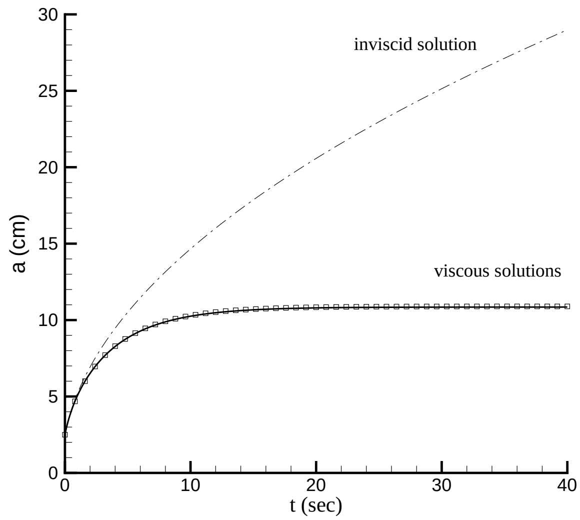

Figure 2: Evolution of the ring radius a with time. The dash-dotted line represents the inviscid solution obtained by putting $\nu = 0$ in (16). The solid line represents the viscous solution of (16) with the initial conditions (19) and (20). The symbol □ represents the viscous solution of (21) with the initial condition (19).

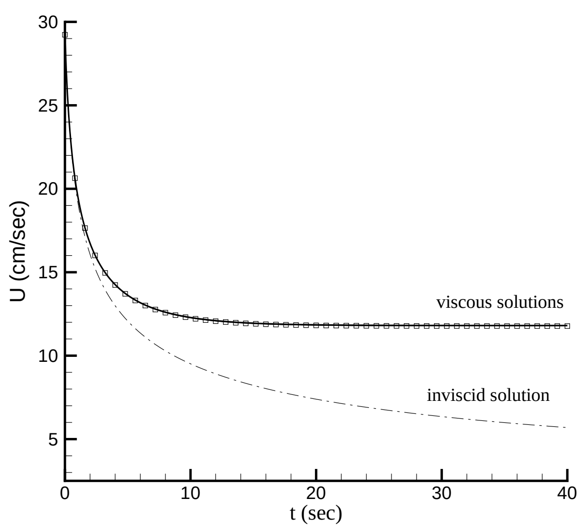

Figure 3: Evolution of the rise velocity U computed from (5) with time. The dash-dotted line represents the inviscid solution obtained by putting $\nu = 0$ in (16). The solid line represents the viscous solution of (16) with the initial conditions (19) and (20). The symbol □ represents the viscous solution of (21) with the initial condition (19).

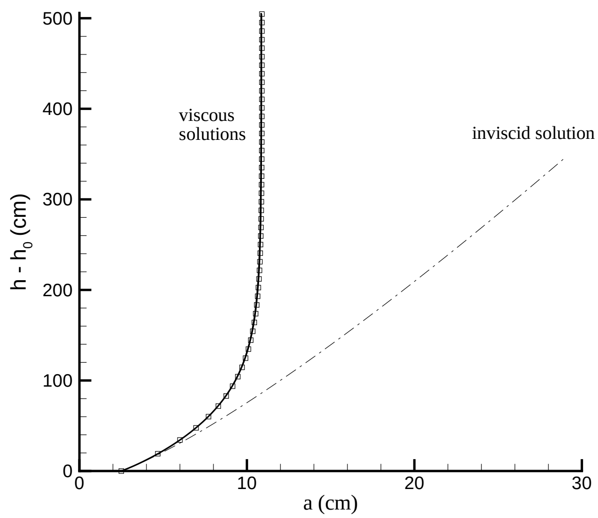

Figure 4: The height of rise $h - h_0$ against the ring radius a. The dash-dotted line represents the inviscid solution obtained by putting $\nu = 0$ in (16). The solid line represents the viscous solution of (16) with the initial conditions (19) and (20). The symbol □ represents the viscous solution of (21) with the initial condition (19).

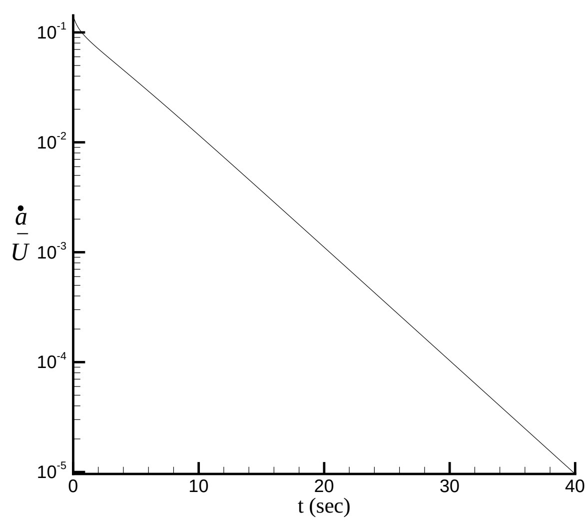

Figure 5: The ratio between the ring expansion velocity $\dot{a}$ and the rise velocity U obtained from the solution of (16).

Acknowledgments

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