Short communication: Bernoulli equation and the competition of elastic and inertial pressures in the potential flow of a second-order fluid

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Abstract

A Bernoulli equation for potential flow of a second order fluid is derived. This equation is used to form an expression for normal extensional stresses at points of stagnation, is which elastic and inertial pressures compete.

Keywords: Bernoulli equation; normal extensional stresses; second order fluid

The stress $\mathbf{T}$ in an incompressible fluid of second grade is given by

$$\mathbf{T} = -p\mathbf{1} + \mu\mathbf{A} + \alpha_1\mathbf{B} + \alpha_2\mathbf{A}^2, \tag{0.1}$$

where $\mathbf{A} = \mathbf{L} + \mathbf{L}^T$ is the symmetric part of the velocity gradient $\mathbf{L} = \nabla\boldsymbol{u}$,

$$\mathbf{B} = \mathbf{A}_{,t} + (\boldsymbol{u}\cdot\nabla)\,\mathbf{A} + \mathbf{A}\mathbf{L} + \mathbf{L}^T\mathbf{A}, \tag{0.2}$$

μ is the zero shear viscosity, $\alpha_1 = -n_1/2$ and $\alpha_2 = n_1 + n_2$ where $[n_1, n_2] = \left[N_1(\kappa^2), N_2(\kappa^2)\right]/\kappa^2$ as $\kappa \to 0$ are the constants obtained from the first and second normal stress difference.

The equations of motion are div$\boldsymbol{u} = 0$ and

$$\rho\left[\boldsymbol{u}_{,t} + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u}\right] = -\nabla P + \mu\nabla^2\boldsymbol{u} + \operatorname{div}\left[\alpha_1\mathbf{B} + \alpha_2\mathbf{A}^2\right], \tag{0.3}$$

where

$$P = p - p_0 - \rho\boldsymbol{g}\cdot\boldsymbol{x} \tag{0.4}$$

is the piezometric pressure, p_0 is a reference pressure and $\boldsymbol{g}$ is gravity.

For potential flow

$$\begin{aligned}
\operatorname{curl}\boldsymbol{u} &= 0,\\
\boldsymbol{u} &= \nabla\phi,\\
\operatorname{div}\boldsymbol{u} = \nabla^2\phi &= 0,\\
(\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} &= \nabla\left(\frac{1}{2}\left|\boldsymbol{u}\right|^2\right),\\
A_{ij} &= 2\phi_{,ij},\\
A_{ij}^2 &= 4\phi_{,il}\phi_{,lj}.
\end{aligned} \tag{0.5}$$

Potential flow of a viscous or viscoelastic liquid is incompatible with the no-slip condition at the boundary of the liquid and solid. It is thought that potential flow is a good approximation under certain circumstances outside a thin boundary layer at the forward side of a body (see for example, Rajeswari and Rathna [1], Beard and Walters [2], Davies [3], Leider and Lilleleht [4]) and dead water region of separated flow at the rear of the body.

Pipkin [5] shows that when (5) holds

$$\mathrm{div}\mathbf{A}^2 = \nabla\gamma^2, \quad \mathrm{div}\mathbf{B} = \frac{3}{2}\nabla\gamma^2, \tag{0.6}$$

where

$$\gamma^2 = \frac{1}{2}\mathrm{tr}\mathbf{A}^2 = 2\phi_{,il}\phi_{,il}. \tag{0.7}$$

After combining $(5)_2$, $(5)_3$, $(5)_4$, $(6)_1$, and $(6)_2$ with (3), we find that

$$\nabla\left[\rho\phi_{,l} + \rho\left|\boldsymbol{u}\right|^2/2 + P - \hat{\beta}\gamma^2/2\right] = 0, \tag{0.8}$$

where $\hat{\beta} = 3\alpha_1 + 2\alpha_2$ is the climbing constant and it is typically positive, $\hat{\beta} > 0$[6]. Hence,

$$\rho\phi_{,t} + \rho\left|\boldsymbol{u}\right|^2/2 + P - \hat{\beta}\gamma^2/2 = C. \tag{0.9}$$

Equation (9) defines the Bernoulli equation for potential flow of a second order fluid. This equation may be simplified for steady flow with uniform streaming U at infinity and $\gamma = p - \rho\boldsymbol{g}\cdot\boldsymbol{x} = 0$ far from the body. In this case

$$\rho\left|\boldsymbol{u}\right|^2/2 + P - \hat{\beta}\rho_{,il}\rho_{,il} = \rho U^2/2. \tag{0.10}$$

Returning next to (1) with (4) and (9), we may eliminate p. Then

$$\sigma_{ij} = -\left[C + \widehat{\beta}\phi_{,lm}\phi_{,lm} - \rho\phi_{,t} - \rho\left|\boldsymbol{u}\right|^2/2\right]\delta_{ij} + 2\left[\mu + \alpha_1\left(\partial_t + \boldsymbol{u}\cdot\nabla\right)\right]\phi_{ij} + 4\left(\alpha_1+\alpha_2\right)\phi_{,il}\phi_{,lj}, \tag{0.11}$$

where

$$\sigma_{ij} = T_{ij} + \rho\mathbf{g}\cdot\mathbf{x}\delta_{ij} \tag{0.12}$$

is the active dynamic stress. In the diagonal coordinates x_1, x_2, x_3 of the frame in which $\phi_{,ij}$ is diagonal

$$[\phi_{,ij}] = \begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}, \tag{0.13}$$

we have

$$\begin{aligned}\begin{bmatrix}\sigma_{11} & 0 & 0\\ 0 & \sigma_{22} & 0\\ 0 & 0 & \sigma_{33}\end{bmatrix} &= -\left[C - \rho\phi_t - \rho\left|\boldsymbol{u}\right|^2/2 + \hat{\beta}\left(\lambda_1^2+\lambda_2^2+\lambda_3^2\right)\right]\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\\ &+2\left[\mu + \alpha_1\left(\partial_t + \nabla\phi\cdot\nabla\right)\right]\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix} + 4\left(\alpha_1+\alpha_2\right)\begin{bmatrix}\lambda_1^2 & 0 & 0\\ 0 & \lambda_2^2 & 0\\ 0 & 0 & \lambda_3^2\end{bmatrix},\end{aligned} \tag{0.14}$$

where

$$\alpha_1 + \alpha_2 = \frac{1}{2}n_1 + n_2 > 0 \tag{0.15}$$

in all viscoelastic fluids known to me.

The case of flow at the stagnation points of a body in steady flow, in an arbitrary direction is of special intrest. The steady streaming past a stationary body is equivalent, under a Galilean transformation, to the steady motion of a body in an otherwise quiet fluid. The potential flow of a fluid near a point $(x_1, x_2, x_3) = (0, 0, 0)$ of stagnation is a purely extensional motion with

$$[\lambda_1, \lambda_2, \lambda_3] = \frac{U}{L}\dot{S}[2, -1, -1], \tag{0.16}$$

where $\dot{S}$ is the dimensionless rate of stretching in the direction x_1, L is the scale of length and

$$[u_1, u_2, u_3] = \frac{U}{L}\dot{S}\left[2x_1, -x_2, -x_3\right]. \tag{0.17}$$

In this case

$$\begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} = \frac{\rho}{2}\left[\frac{U^2}{L^2}\dot{S}^2\left(4x_1^2 + x_2^2 + x_3^2\right) - U^2\right]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$+\mu\frac{U}{L}\dot{S}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + 2\frac{U^2}{L^2}\dot{S}^2\begin{bmatrix} -\alpha_1 + 2\alpha_2 & 0 & 0 \\ 0 & -7\alpha_1 - 4\alpha_2 & 0 \\ 0 & 0 & -7\alpha_1 - 4\alpha_2 \end{bmatrix} \tag{0.18}$$

At the stagnation point itself

$$\sigma_{11} = -\frac{\rho}{2}U^2 + 2\mu\dot{S} + 2\left(2\alpha_2 - \alpha_1\right)\frac{U^2}{L^2}\dot{S}^2. \tag{0.19}$$

Since $\alpha_1 < 0$, $2\alpha_2 - \alpha_1 = \frac{5}{2}n_1 + 2n_2 > 0$, the normal stress term in (19) is positive, independent of the sign of $\dot{S}$, but $2\mu\dot{S}$ is negative at the front side of a falling body and is positive at the rear. This is a new manifestation of the competition between inertia and normal stress, which I believe plays a major role in recently observed flow induced anisotropy [7]. This causes long bodies to float broadside on when the ratio of inertial pressure to normal (extensional) stress is large, and long-side on in the other case. The same flow induced anisotropy causes suspensions of solid spheres to develop cross strcam structure when the ratio is high, and to hook together in vertical chains when it is low.

Diensionless groups may be formed from the ratios of inertia $\rho U^2/2$, viscosity $2\mu U/L$ and normal-extensional stresses $(5n_1 + 4n_2)U^2/L^2$. We could again speak of an inertial radius (Joseph [6]) for the competition between inertial and normal-extensional stress with inertia dominant when $L > L_c$ and normal stress dominant when $L < L_c$ where

$$L_c^2 \approx \frac{10n_1 + 8n_2}{\rho}.$$

is a material property. Riddle et al. [8] discovered that if the initial separation of two spheres settling along their line of centers in a viscoelas-tic flued is larger than a certain critical separation the spheres will diverge, whereas if it is smaller than this separation they will converge. This is consistent with the notion that the critical separation is the inertial radius L_c.

Every potential flow is a solution of the equations of motion for a fluid of second grade with stresses given by (11). Such solutions do not generally satisfy the condition of no-slip at solid boundaries.

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