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Lift and multiple equilibrium positions of a single particle in Newtonian and Oldroyd-B fluids

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Received 6 August 2004; received in revised form 9 December 2004; accepted 14 December 2004
Available online 23 February 2005

Abstract

A heavy particle is lifted from the bottom of a channel in a plane Poiseuille flow when the Reynolds number is larger than a critical value. In this paper we obtain correlations for lift-off of particles in Oldroyd-B fluids. The fluid elasticity reduces the critical shear Reynolds number for lift-off. The effect of the gap size between the particle and the wall, on the lift force, is also studied. A particle lifted from the channel wall attains an equilibrium height at which its buoyant weight is balanced by the hydrodynamic lift force. Choi and Joseph [Choi HG, Joseph DD. Fluidization by lift of 300 circular particles in plane Poiseuille flow by direct numerical simulation. J Fluid Mech 2001;438:101–128] first observed multiple equilibrium positions for a particle in Newtonian fluids. We report several new results for the Newtonian fluid case based on a detailed study of the multiple equilibrium solutions, e.g., we find that at a given Reynolds number there are regions inside the channel where no particle, irrespective of its weight, can attain a stable equilibrium position. This would result in particle-depleted zones in channels with Poiseuille flows of a dilute suspension of particles of varying densities. Multiple equilibrium positions of particles are also found in Oldroyd-B fluids. All the results in this paper are based on 2D direct numerical simulations.

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1. Introduction

Fluid-particle suspension transport occurs in diverse applications e.g. sand transport in fractured reservoirs, slurry flows, cleaning particles from surfaces, removal of drill cuttings in horizontal drill holes etc. The hydrodynamic lift force that balances the buoyant weight of the particles plays a central role in the transport of heavy (i.e. density greater than that of the suspending fluid) particles in channels. In this case the particles are advected after resuspension by the lift force. A force experienced by a particle moving through a fluid with circulation (or shearing motion for a viscous fluid) shall be referred to as the lift force. Models for lift forces in mixtures, to be used as predictive engineering tools, are much less well developed than e.g. models for drag.

Joseph [8] proposed that ideas analogous to the Richardson and Zaki [13] correlation must come into play in problems of slurries, where the particles are fluidized by lift rather than by drag. The problems of resuspension by lift may be decomposed into two separate types of study: (1) single particle studies in which the factors that govern lifting of a heavier-than-liquid particle off a wall by a shear flow are identified and (2) many particle studies in which cooperative effects on lift-off are investigated. The primary motivation of this work is to contribute toward a better understanding of the lift force on a single particle and to identify the nature of the correlations describing it. These studies are expected to have direct bearing on a systematic effort to develop a formula for lift on a particle in dilute and concentrated suspensions. Such a methodology is not unknown and has been remarkably successful in the modeling of drag—a formula that is extensively used in industrial applications. The formula for drag is attributed to Richardson and Zaki [13] who obtained an expression for the fluidization velocity of many particles in terms of the average fluid fraction and the fluidization velocity of a single particle. The extension of this approach to model lift is described by Patankar et al. [12].

Different analytical expressions for the lift force on a single particle in Newtonian fluids can be found in literature. They are based on perturbing Stokes flow with inertia or on perturbing potential flow with a little vorticity. In particular, Schonberg and Hinch [14]; Hogg [3] and Asmolov [1] analytically studied the inertial migration of spherical particles in Poiseuille flows. The effect of curvature of the unperturbed velocity profile was found to be important. The domain of parameters such as the Reynolds numbers for which these analytic expressions are applicable is restricted. The perturbation analyses are of considerable value because they are analytic and explicit but they are not directly applicable to engineering problems like proppant transport, removal of drill cuttings, sediment transport or even lift-off of heavy single particles.

In some applications e.g. proppant transport in hydraulic fractures, the suspending fluid can be viscoelastic. The lift force on a sphere in a linear shear flow of a second-order fluid was studied by Hu and Joseph [4]. Their analysis is valid for slow and slowly varying flows. They found that the normal stress effect enhances the lift on the particles.

Patankar et al. [11] considered a particle heavier than the fluid driven forward on the bottom of a channel by a plane Poiseuille flow. After a certain critical Reynolds number the particle rises from the wall to an equilibrium height at which the buoyant weight just balances the upward thrust from the hydrodynamic force. Multiple equilibrium positions for a heavy particle at the same Reynolds number was first detected by Choi and Joseph [2]. It was explained by Patankar et al. [11] to be due to the presence of a turning point 'bifurcation' of the equilibrium solution. A correlation for the critical shear Reynolds number for lift-off was obtained for a particle in a Newtonian fluid.
The equilibrium height of a single particle in horizontal channel flows is important for the transport of dilute suspensions of heavy particles. At the equilibrium height the hydrodynamic lift force just balances the buoyant weight of the particle. During the transport process, particles in a dilute suspension tend to migrate to the stable equilibrium height. The experiments of Segré and Silberberg [15] have shown this for the Poiseuille flow of a dilute suspension of neutrally buoyant spheres in a pipe. They found that at low Reynolds numbers, the spheres in a dilute suspension migrate to an equilibrium position of 0.6 times the pipe radius. Their experiments had a big influence on studies of the fluid mechanics of migration and lift. The knowledge of equilibrium positions of single particles is not only useful in understanding the migration in dilute suspensions but can also lead to further insights into the modeling of many particle lift-off.

The governing equations and the parameters of the problem will be presented in Section 2. In Section 3, we will study the critical condition for the lift-off of a circular particle close to the wall in a plane Poiseuille flow of an Oldroyd-B fluid. The results for the Oldroyd-B fluid case will be compared to the corresponding correlation for the Newtonian case reported by Patankar et al. [11]. In Section 4.1 we will study the multiple equilibrium positions of a single particle in Poiseuille flows of Newtonian fluids. Correlations for the turning points in the equilibrium diagram will be presented in Section 4.2. In Section 4.3 we will investigate the effect of the channel width on the lift force and the equilibrium diagram. Results on multiple equilibrium positions in Oldroyd-B fluids will be presented in Section 5 and conclusions in Section 6.

2. Governing equations and the parameters of the problem

A brief discussion on the driving mechanism, the applied pressure gradient, is in order before we present the governing equations. The two-dimensional computational domain for our simulations is shown in Fig. 1. We perform simulations in a periodic domain. The applied pressure gradient is given by $-p$. We split the pressure as follows:

$$ P = p + \rho f g \cdot x - \bar{p} e_z \cdot x = -\nabla p + \rho f g = -\nabla p + \bar{p} e_z, $$

where $P(x, t)$ is the pressure, $\rho_f$ is the fluid density, $g = -ge_y$ is the acceleration due to gravity, $x$ is the position vector at a point, and $e_x$ and $e_y$ are the unit vectors in the $x$- and $y$-directions, respectively. We solve for the 'dynamic' pressure $p$ in our simulations. The external pressure gradient term then appears as a body force-like term in the fluid and particle equations, as seen below.

![Computational domain for the lift-off of a single particle in plane Poiseuille flows.](image)
The governing equations in non-dimensional form are:

\[ \nabla \cdot \mathbf{u} = 0, \]

\[ R \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = -\nabla p + 2 \frac{d}{W} \mathbf{e}_x + \nabla \cdot \mathbf{T}, \]

\[ \mathbf{T} = \mathbf{A}; \quad \text{for a Newtonian fluid}, \]

\[ \mathbf{T} + D \mathbf{e} \mathbf{T} = \left( \mathbf{A} + \frac{\lambda_2}{\lambda_1} D \mathbf{e} \mathbf{A} \right); \quad \text{for an Oldroyd-B fluid}, \]

\[ \frac{\rho_p}{\rho_r} \frac{d \mathbf{U}_p}{dt} = 2 \frac{d}{W} \mathbf{e}_x + \frac{4}{\pi} \oint [-p \mathbf{I} + \mathbf{T}] \cdot \mathbf{n} d\Gamma, \]

\[ \frac{\rho_p}{\rho_r} \frac{d \Omega_p}{dt} = \frac{32}{\pi} \oint (\mathbf{x} - \mathbf{X}) \times \left( [-p \mathbf{I} + \mathbf{T}] \cdot \mathbf{n} \right) d\Gamma, \]

where \( \mathbf{u}(x,t) \) is the fluid velocity, \( \rho_p \) is the particle density, \( \mathbf{T} \) is the extra-stress tensor, \( \mathbf{A} = (\mathbf{V} \mathbf{u} + \mathbf{V} \mathbf{u}^T) \) is two times the deformation-rate tensor, \( \lambda_1 \) and \( \lambda_2 \) are the constant relaxation and retardation times, respectively, \( \mathbf{U}_p \) is the translational velocity of the particle in the axial direction, \( \Omega_p \) is the angular velocity of the particle, \( \mathbf{X} \) is the coordinate of the center of mass of the particle and we consider circular particles of diameter \( d \). The non-dimensional parameters will be presented shortly. Eq. (2) and the corresponding initial and boundary conditions define an initial boundary value problem that can be solved by direct numerical simulation.

The gravity term is not present in the fluid equation because it is balanced by the hydrostatic pressure component (see Eq. (1)). The gravity term is expected in the y-momentum equation (not listed in Eq. (2)) of the particle but we do not solve this equation because the particle height is fixed. Only the axial and angular motion equations of the particle (listed in Eq. (2)) are solved in our simulations. We refer to these as constrained simulations (explained below) and were introduced by Patankar et al. [11].

In constrained simulations the particle is free to rotate and translate in the axial \((x-)\) direction. The height, \( h \), of the particle center from the bottom wall of the channel is fixed so that it does not translate in the transverse direction. The gap \( \delta \) between the particle and the bottom wall is \( h - d/2 \). There is no external body force in the axial direction and no external torque is applied. The particle is initially at rest. The simulation in the periodic domain is started by imposing the external pressure gradient in the form of the body force-like term. The particle begins free rotation and translation in the axial direction. It eventually reaches a state of steady motion. In steady condition the particle translates in the axial direction at a constant velocity and rotates at a constant angular velocity. These velocities are such that there is no net hydrodynamic drag (force along the axial direction) or torque. Since the particle height is fixed i.e. it does not freely translate in the transverse direction, the hydrodynamic lift force (in the transverse direction), \( L \), on the particle is not zero in general and is obtained as a part of the solution.

The steady state translational and angular velocities as well as the hydrodynamic lift force obtained from the constrained simulations are independent of the particle density used in our calculations. Note that the particle acceleration term which has the particle density as its coefficient drops out at steady state (see Eq. (2)). Only the transient, which is not the focus of this paper, depends on the choice of the particle density used in the simulations but the final steady state
is the same. Constrained simulations are very convenient to study the lift force and the equilibrium height of a particle. It not only allows the identification of stable equilibrium heights but also of unstable equilibrium positions not accessible from unconstrained (or dynamic) simulations. This will be evident in Section 4.1.

The hydrodynamic lift force \( L \), obtained from the constrained simulation, could be considered to balance the buoyant weight of a particle of density \( \rho_p \) given by

\[
\rho_p = \rho_t + \frac{L}{V_p g},
\]

where \( V_p \) is the volume per unit length of the particle. The prescribed height, \( h \), can then be considered to be the equilibrium position of this particle. We follow this interpretation while presenting our results.

The parameters in this problem at steady condition are [11]:

\[
e = \frac{\delta}{d}, \quad \text{non-dimensional gap},
\]

\[
R = \frac{\rho_t V d}{\eta} = \frac{\rho_t \dot{\gamma}_w d^2}{\eta} = \frac{\rho_t W d^2}{2\eta^2 - \bar{p}}, \quad \text{shear Reynolds number},
\]

\[
R_G = \frac{\rho_t d^3 L}{\eta^2 V_p} = 4 \frac{\rho_t d L}{\pi \eta^2} = \frac{4 \rho_t d (\rho_p - \rho_t) V_p g}{\pi \eta^2}, \quad \text{non-dimensional lift or buoyant weight},
\]

\[
\frac{\bar{p} d^2}{\eta V} \approx \frac{\bar{p} d}{\eta \dot{\gamma}_w} = 2 \frac{d}{W}, \quad \text{aspect ratio},
\]

\[
E = DeR = \frac{\dot{\lambda}_1 V}{d \cdot R} = \frac{\dot{\lambda}_1 \eta}{\rho_t d^2}, \quad \text{elasticity number and}
\]

\[
\frac{\dot{\lambda}_2}{\dot{\lambda}_1}, \quad \text{ratio of retardation and relaxation times},
\]

where \( De \) is the Deborah number and a dimensionless description of the governing equations is constructed by introducing the following scales: the particle size \( d \) for length, \( V \) for velocity, \( d/V \) for time, \( \eta V/d \) for stress and pressure and \( V/d \) for the angular velocity of the particle, where \( \eta \) is the viscosity of the fluid. We have chosen \( V = \dot{\gamma}_w d \), where \( \dot{\gamma}_w \) is the shear-rate at the wall (in the absence of the particle) as shown in Fig. 1. The elasticity of the fluid increases with the elasticity number whereas \( \dot{\lambda}_2/\dot{\lambda}_1 = 1 \) corresponds to a Newtonian fluid and \( \dot{\lambda}_2/\dot{\lambda}_1 = 0 \) corresponds to a highly elastic Maxwell fluid.

In our simulations the periodic length, \( l \), of the channel is chosen large enough so that the solution is only weakly dependent on its value. In Fig. 2 the hydrodynamic lift force on a particle in periodic and non-periodic domains is compared. The agreement is good.

An arbitrary Lagrangian–Eulerian (ALE) moving mesh technique with the EVSS (elastic-viscous-stress-split) scheme for the Oldroyd-B constitutive model is used to solve the governing equations. More details of this numerical scheme are given by Hu [5] and Hu et al. [6].
The dimensionless lift on the particle depends on the parameters listed above:

$$R_G = f \left( R, \frac{W}{d}, e, E, \frac{\lambda_2}{\lambda_1} \right).$$

(4)

3. Lift-off of a single particle in an Oldroyd-B fluid

Krishnan and Leighton [10] calculated the lift force on a smooth sphere rotating and translating in a simple shear flow in contact with a rigid wall. Hu and Joseph [4] extended their analysis to second-order fluids. Their results were valid at low Reynolds numbers. The non-dimensional lift $R_G$ for a particle in an Oldroyd-B fluid is given by [4]:

$$R_G = \frac{3}{2\pi} \left( 1.755 \lambda_2^2 + 0.1365 \lambda_1^2 - 0.019 R_U R_Q - 4.522 R_U R + 0.303 R_Q R + 2.314 R^2 \right)$$

$$+ \frac{24}{5 \eta} E \left( 1 - \frac{\lambda_2}{\lambda_1} \right) \left( \lambda_2^2 + 0.25 \lambda_2^2 - 0.25 R_U R_Q \right),$$

(5)

$$R_U = \frac{\rho_f U d}{\eta} \text{ and } R_Q = \frac{\rho_f \Omega d^2}{\eta}.$$

The above expression is valid in the limit of slow and slowly varying flows so that the second-order fluid expansion is valid.

For a freely translating and rotating sphere, $R_U$ and $R_Q$ are functions of the flow parameters like $R$, the gap size etc. The above expression was obtained for semi-infinite domains. Hence $W/d$ is not a parameter. Eq. (5) is in agreement with the general form in Eq. (4).

The expression for the Newtonian case is obtained by substituting $E = 0$ in Eq. (5). The resulting expression is valid in the limit of zero gap size. A heavy particle freely translating and rotating in contact with a plane wall in simple shear flow of a Newtonian fluid is lifted from the wall and
suspended in the fluid if the shear Reynolds number $R$ is greater than a critical value. Beyond the critical shear Reynolds number the particle rises from the wall to an equilibrium height at which the buoyant weight just balances the upward thrust from the hydrodynamic force. In a Newtonian fluid the case of zero separation distance corresponds to an infinite drag force due to the logarithmic singularities in the lubrication equations for drag and torque. This results in zero translational and rotational velocities of the particle [10]. For a particle in a viscoelastic fluid the elastic component of the lift force is also singular when the gap between the sphere and the wall approaches zero. This is an important qualitative feature that differentiates the lift force in a Newtonian and a viscoelastic fluid.

In practical applications the particle acquires some finite separation distance from the wall due to the presence of surface roughness. This eliminates the lubrication singularity in a Newtonian fluid [10,16]. The additional non-hydrodynamic frictional force due to the surface roughness does not significantly affect the lift force on the particle within a reasonable range of the coefficient of friction for particles in a Newtonian fluid [10].

Patankar et al. [11] investigated the lift-off of a circular particle in a plane Poiseuille flow of a Newtonian fluid. Their results were limited to two-dimensional cases but not restricted to low Reynolds numbers. In simulations the gap between the particle and the wall cannot be zero [6]. Following Krishnan and Leighton [10] the minimum gap size (or surface roughness) was set to 0.001$d$. The additional frictional force due to surface roughness was assumed to be zero. The critical Reynolds number was defined as the minimum shear Reynolds number required to lift a particle to an equilibrium height greater than 0.501$d$. Patankar et al. [11] reported that the effect of the channel width on the critical shear Reynolds number for $W/d > 12$ was small. They obtained a correlation for $R_c$ as a function of $R$. Here we present results for a particle in an Oldroyd-B fluid. To facilitate comparison we use $W/d = 12$ and $\varepsilon = 0.001$ in our simulations.

Fig. 3a and b show the plot of $R_c$ vs. the critical shear Reynolds number for lift-off at different values of the elasticity number and $\lambda_2/\lambda_1$, respectively. The fluid elasticity enhances the lift on the particle at given Reynolds number. The smaller ratio of retardation and relaxation times increases the lift on the particle. The data from the simulations can be represented by a power law equation given by $R_c = a R^n$, where the values of $a$ and $n$ are listed in the figures. We observe that the slopes, $n$, for Newtonian and Oldroyd-B fluids are different. The prefactor $a$ also changes as $\lambda_2/\lambda_1$ changes.

In a three-dimensional low Reynolds number case, Eq. (5) predicts that the effect of elasticity is represented by $E(1 - \lambda_2/\lambda_1)$. Fig. 4a shows the plot of $a$ vs. $E(1 - \lambda_2/\lambda_1)$. We observe that $a$ varies linearly with respect to $E(1 - \lambda_2/\lambda_1)$. Note that Eq. (5) predicts similar behavior. The power law equation for $R_c$ can be rewritten by using the equation for $a$, given in Fig. 4a, as:

$$R_c = 102.08 E \left(1 - \frac{\lambda_2}{\lambda_1}\right) R^n + 1.1863R^n \quad \text{for} \quad W/d = 12, \ \varepsilon = 0.001.$$  \hspace{1cm} (6)

This correlation is similar to the theoretical expression for a three-dimensional case in Eq. (5). Fig. 5 shows the variation of $n$ vs. $E(1 - \lambda_2/\lambda_1)$. It should be noted that $n$ is close to 2 (consistent with the low Reynolds number approximation) for larger values of $E(1 - \lambda_2/\lambda_1)$. The variation in the value of $n$ is less pronounced compared to the variation of $a$. 
A qualitative comparison between Eqs. (5) and (6) suggests that $R_G$ and $R_D$ depend on $R$ but is only weakly dependent on the elasticity parameter $E(1 - \lambda_3/\lambda_1)$. This is verified in Fig. 5a and b. The data from Newtonian as well as Oldroyd-B cases fall on the same curve. The leading order term of the fitted curve is linear thus indicating that the inertial effects are not dominant. These results imply that the non-dimensional translational and angular velocities of the particle, at a given location in the channel and at a given Reynolds number, are almost the same in Newtonian and Oldroyd-B cases.
The Tanner–Pipkin theorem states that the Newtonian and second-order fluid flow fields are identical in two-dimensions in the zero Reynolds number limit. The Oldroyd-B fluid can be represented by the second-order fluid model for slow and slowly varying flows. The result in Fig. 5 although not limited to zero Reynolds is similar to the conclusions of the Tanner–Pipkin theorem. For the parameters we have considered, we see that the flow field in an Oldroyd-B case is not very different from the Newtonian case at the same Reynolds number—hence the effect of elasticity on $R_U$ and $R_G$ is not significant.

In the above simulations the critical shear Reynolds number for lift-off is defined by an equilibrium height corresponding to $\varepsilon = 0.001$. Eqs. (4) and (5) predict that in general $R_G$ is a function of...
Fig. 5. (a) The plot of $R_U$ vs. $R$. Data from Newtonian as well as Oldroyd-B cases fall on a single curve ($W/d = 12$, $\epsilon = 0.001$). (b) The plot of $R_{\Omega}$ vs. $R$. Data from Newtonian as well as Oldroyd-B cases fall on a single curve ($W/d = 12$, $\epsilon = 0.001$).

The gap size. Plots similar to Fig. 3 for a Newtonian and an Oldroyd-B fluid can be obtained for different gap sizes. Our simulations revealed that the slope $n$ does not significantly change for gap sizes ranging from $\epsilon = 0.001$ to $\epsilon = 0.022$. We chose $W/d = 12$ as in Fig. 3. The value of $n$ was close to 2 for Oldroyd-B fluids and 1.38 for Newtonian fluids. However, the value of $\alpha$ changed with the gap size.

Fig. 6 compares the variation of $\alpha$ vs. $\epsilon$ between a Newtonian and a particular Oldroyd-B fluid ($\lambda_2/\lambda_1 = 0.125$, $E = 0.05$). It is seen that the value of $\alpha$ increases rapidly as the gap size tends to zero in an Oldroyd-B fluid; in qualitative agreement with the theoretical predictions of Eq. (5). Such behavior is not observed for the Newtonian case.
4. Multiple equilibrium positions in Newtonian fluids

4.1. Equilibrium diagrams

The parameters chosen are $l/d = 22$, $W/d = 12$ and $d = 1$ cm. The fluid density is 1 g/cc and its viscosity is 1 poise. From Eq. (4), the non-dimensional hydrodynamic lift force $R_G$ depends on $\varepsilon$ and the shear Reynolds number.

Fig. 7 shows the plot of $R_G$ as a function of the height of the particle center at different values of shear Reynolds numbers. Simulations were performed for Reynolds numbers ranging from 1 to 250. Fig. 7 shows typical plots for $R$ up to 15. The magnitude of $R_G$ increases significantly over the entire range of Reynolds numbers. The values of $R_G$ are anti-symmetric with respect to the centerline of the channel. The non-dimensional lift force $R_G$ shows a non-monotonic behavior with respect to the particle position in the channel, which results in multiple stable equilibrium positions for a particle.

Plots like those in Fig. 7 can be used to obtain the equilibrium height diagram of a particle of given density [11]. Consider a particle of density 1.01 g/cc i.e. $R_G = 9.81$. The equilibrium heights for this particle (refer Eq. (3) and the corresponding discussion), at a given shear Reynolds number, are identified as the points of intersection between the curve of $R_G$ vs. $h/l$ and $R_G = 9.81$ (see Fig. 8). The intersection points where the slope of $R_G$ vs. $h/l$ curve is positive are unstable equilibrium points whereas a negative slope represents a stable equilibrium point. This is indeed confirmed by dynamic (unconstrained) simulations of Choi and Joseph [2].

Fig. 9 shows the equilibrium height of a particle with $R_G = 9.81$ at different values of $R$. This is the equilibrium diagram of this particle in a channel with $W/d = 12$. An incomplete equilibrium diagram was first obtained by Choi and Joseph [2] who performed dynamic simulations where
the particle was free to move in the transverse direction as well. They were unable to plot the unstable branch of the equilibrium height since it could not be obtained from the dynamic simulations. We see a turning point solution of the equilibrium height in Fig. 9. A similar plot showing equilibrium positions in the bottom half of the channel was first presented by Patankar et al. [11]. They explained the different equilibrium positions by analyzing the pressure and shear stress contributions to the lift force on a particle.

In Fig. 9 we also observe a stable and an unstable branch of equilibrium positions in the top half of the channel at shear Reynolds numbers of 100 and more. This is due to a turning point in the upper half of the channel. These branches were not previously reported by Patankar et al. [11] since they performed simulations up to $R = 80$. This result is new and implies that a heavy particle can achieve equilibrium in the upper half of the channel.

Similarly, equilibrium diagrams of particles of different densities can be obtained by following the procedure indicated in Fig. 8. Different types of equilibrium diagrams are possible. Fig. 10
shows the equilibrium diagram of a particle with $R_G = 3.14$. Unlike the equilibrium diagram in Fig. 9, there are no turning points in the bottom half of the channel. There is one turning point in the top half of the channel. This is another type of equilibrium diagram for a heavy particle.

Fig. 10 shows the equilibrium diagram of a neutrally buoyant particle. There is one stable equilibrium position symmetrically located on either side of the channel centerline. The equilibrium position at the channel centerline is unstable. Segré and Silberberg [15] found that at low Reynolds number Poiseuille flows, the equilibrium position of a sphere from the pipe axis was 0.6 times the pipe radius. In Fig. 11 we see that the equilibrium position is a function of the shear
Reynolds number, albeit in two dimensions. At \( R = 1 \) the equilibrium position is 0.49 times the channel half width. As the Reynolds number is increased the equilibrium position moves towards the channel centerline. The largest Reynolds we could simulate without convergence problems was \( R = 250 \). At this value the equilibrium position is still changing with the Reynolds number (Fig. 11). Prediction of the trend at higher Reynolds numbers is not straightforward. The equilibrium height may have an asymptotic value or Hopf bifurcation (possibly due to vortex shedding) may also be observed. In a three-dimensional situation turbulence may set in at these high Reynolds numbers. Turbulence suppression is likely in our two-dimensional simulations at high Reynolds numbers. Unlike uniform flow past a cylinder at high Reynolds numbers, we do not observe vortex shedding for a cylinder in Poiseuille flow up to the maximum shear Reynolds number in our simulations. This is primarily due to small slip velocity between the particle and the fluid. Small slip velocities were also noted by Huang and Joseph [7], Patankar et al. [11] and Joseph and Ocando [9].

Figs. 9–11 represent the primary types of equilibrium diagrams possible for particles in Poiseuille flows. Figs. 9 and 10 represent equilibrium diagrams for heavy particles. Equilibrium diagrams from light (i.e. density less than that of the suspending fluid) particles are similar; e.g. the equilibrium diagram for a light particle with \( R_G = -3.14 \) can be obtained from Fig. 10 by simply turning the figure upside down. This follows from the anti-symmetric nature of the lift force with respect to the channel centerline (Fig. 7).

Fig. 12 shows the equilibrium diagrams for different values of \( R_G \). The turning points in the top and bottom halves of the channel occur at higher Reynolds numbers as the value of \( R_G \) increases. The equilibrium positions of different particles differ only slightly at higher Reynolds numbers. All the equilibrium positions, stable or unstable, shift towards the channel centerline at high Reynolds numbers.

At a given Reynolds number, a heavy particle, no matter what its weight, will not migrate to regions of unstable equilibrium or of negative lift. In Fig. 13, the regions marked as \((A')\), \((A'')\) and \((A'''\) along the axis represent zones where a heavy particle can be in stable equilibrium at
Fig. 12. Equilibrium diagrams at different values of $R_G$. Both, the stable and the unstable branches are shown.

Fig. 13. Stable and unstable equilibrium regions at $R = 17.5$ and $W/d = 12$. Zones $(A', A'', A''')$ represent the stable equilibrium region for a heavy particle. Zone (B) represents the unstable equilibrium region where no particle attains a stable height. Zone (C) represents the stable equilibrium region for a light particle.

$R = 17.5$. The regions marked as (B) and (C) represent zones where a heavy particle can never be in stable equilibrium at $R = 17.5$. Similar information, for heavy particles, at different Reynolds numbers is plotted collectively in Fig. 14. The zones where no heavy particle can attain stable equilibrium (no-equilibrium zones) are marked as shaded regions, which are obtained e.g. from
Fig. 14. The shaded regions represent no-equilibrium zones (i.e. an unstable equilibrium or a negative lift region) for a heavy particle ($W/d = 12$). The data are obtained for $1 \leq R \leq 250$ from Fig. 8.

(B) and (C) regions in Fig. 13 at a given Reynolds number. A similar plot can be obtained for light particles by turning Fig. 14 upside down. The plots for heavy and light particles can be combined to give Fig. 15. Fig. 15 identifies no-equilibrium zones that are common to both heavy and light particles. It can be easily seen that, at a given Reynolds number, the no-equilibrium zones common to both heavy and light particles are indeed the unstable equilibrium zones marked by region (B) in Fig. 13. Although the actual equilibrium height of a particle depends on the value of $R_c$ for

Fig. 15. The unstable equilibrium zones for heavy or light particles at various Reynolds numbers ($W/d = 12$).
that particle, no heavy or light particle will migrate to the unstable equilibrium zones marked in Fig. 15.

Fig. 15 has important implications for the Poiseuille flows of dilute suspensions of particles of different densities. In such flows the particles will migrate to their respective stable equilibrium positions. Fig. 15 implies that, at a given Reynolds number, there will be regions in the channel to which no particles will migrate, irrespective of their density. The results in Fig. 15 are limited to \( Wld = 12 \). In Section 4.3, we will extend this diagram to account for different values of \( Wld \).

4.2. Correlations for turning points

The turning points are the key features in the equilibrium diagram that can help identify the number of stable equilibrium positions of a particle in a channel. In this section we obtain correlations for the turning points in the equilibrium diagram.

To obtain correlations, the turning points are identified as maxima or minima in the plot of \( R_G \) vs. \( hld \) at a given \( R \) (see Fig. 8). It is easy to verify that the plots of \( R_G \) vs. \( hld \) at different values of \( R \) are equivalent to the previously defined ‘equilibrium diagrams’, which are plots of \( hld \) vs. \( R \) at different values of \( R_G \). In Fig. 16, we plot the values of \( R_G \) at three turning points (ref. Fig. 8) as a function of \( R \) for \( R > 10 \). These three turning points are sufficient to identify all the turning points in the channel by using the anti-symmetry property of the plot of \( R_G \). For \( R < 10 \), the first minimum and the first maximum (see Fig. 8) did not exist and led to a monotonous plot of \( R_G \) in that region. This is evident in Fig. 7.

It is seen that power laws closely describe the data for each of the turning points. The power law correlations are given in Fig. 16. These correlations can be used to identify the number of stable equilibrium positions of a particle. To this end consider the case of a heavy particle with given values of \( R_G \) and \( R \) in a channel with \( Wld = 12 \). This case can be identified by a point in the

![Diagram](image_url)

Fig. 16. Power law correlations for the turning points in the equilibrium diagram (\( Wld = 12 \)).
$R_G$–$R$ parameter space in Fig. 16. There are four possibilities: (1) If the point is above the power law line of the first maximum then there will be one stable equilibrium position near the bottom wall. (2) If the point lies between the power law lines of the first maximum and the first minimum then there will be two stable equilibrium positions—one near the bottom wall and the other closer to the center line in the bottom half of the channel. (3) If the point is between the first minimum and the second maximum lines then there will be one stable equilibrium in the bottom half of the channel. (4) Lastly, if the point is below the second maximum line then there will be two stable positions—one in the bottom half of the channel and the second in the top half of the channel. The above arguments are clearer by referring to Fig. 8. Extension of the arguments to light particles is straightforward.

The power correlations thus lead to the prediction of the range of Reynolds numbers or pressure gradients that will give either two equilibrium positions or one equilibrium position for a given particle. The actual equilibrium position is not obtained from these correlations.

4.3. The effect of channel width

In this section, we study the effect of the channel width. Simulations at two more channel widths, $W/d = 6$ and $W/d = 24$, are performed. The parameters used here are the same as those in Section 4.1; only the value of $W/d$ is changed.

Plots for non-dimensional lift $R_G$ vs. $h/d$ for $W/d = 6$ and 24 were obtained from simulations. A typical plot for $W/d = 6$ is shown in Fig. 17. $R_G$ is plotted only for the lower half of the channel; the plot of $R_G$ in the other half of the channel being anti-symmetric with respect to the lower half. The minimum height of center of the particle is $0.75d$. We had convergence difficulties for particle heights less $0.75d$. Even if the first minimum point, observed in Fig. 7 for $W/d = 12$, is not obtained in Fig. 17 for $W/d = 6$, there probably will be a minimum point between $h/d = 0.5$ and

Fig. 17. Plot $R_G$ vs. the height of the particle center from the bottom wall of the channel with $W/d = 6$ and $R = 100$–250.
0.75 for given values of $R$. For $W/d = 6$, we did simulations for $R$ from 10 to 250 and, for $W/d = 24$ for $R$ from 1 to 100. Beyond $R = 100$, for $W/d = 24$, we encountered convergence problems because the channel Reynolds number (defined in Eq. (7) below) is high.

In Fig. 18 we show $R_G$ as a function of $h/W$ at $R = 10$ for three different values of $W/d$. We observe that there are two stable equilibrium heights for certain values of $R_G$ when $W/d = 12$ and 24, but not so for $W/d = 6$. Note that we use $h/W$ instead of $h/d$ to compare the qualitative nature of the plot of $R_G$. Clearly, the qualitative nature of the plot of $R_G$ at the same value of $R$ is quite different for the three widths.

We investigate whether the channel Reynolds number is a more appropriate parameter to compare the plots. The channel Reynolds number $R_{ch}$ is based on the width of the channel and the mean fluid velocity in the absence of the particle. $R$ and $R_{ch}$ are related as follows:

$$R_{ch} = \frac{\rho_t V_m W}{\eta} = \frac{\rho_t W}{\eta} \frac{W^2}{12\eta\bar{P}} = \frac{\rho_t Wd^2}{6d^2} \frac{W^2}{2\eta\bar{P}} = \frac{W^2}{6d^2} R,$$  

(7)

where $V_m$ is the mean velocity of Poiseuille flow in the absence of the particle.

Fig. 19a shows the plot of $R_G$ vs. $h/W$ for $R_{ch} = 60$. Two channel widths, $W/d = 6$ and $W/d = 12$, are shown. For $W/d = 12$ we plot a scaled (13 times) value of $R_G$. The scaling, chosen arbitrarily through trial and error, helps in comparing the qualitative nature of the plot of $R_G$. We see that in the region away from the wall, the shape of the lift curve depends on $h/W$ and $R_{ch}$ only—thus the particle size play no role. The magnitude of $R_G$ does depend on $W/d$ as evidenced by the scaling required in the comparison. Fig. 19b–f shows that the above conclusion is remarkably valid over the range of parameters we considered.

In the regions close to the walls, both the shape and the magnitude of the lift curve depend on parameters that include the effect of the particle size. This is consistent with the results in Section 3.

![Graph](image)

Fig. 18. Comparison of $R_G$ as function of $h/W$ for $W/d = 6$, 12 and 24 at $R = 10$. 

**Notes:**

- $V_m$ refers to the mean velocity of Poiseuille flow in the absence of the particle.
- $R_{ch}$ is the channel Reynolds number based on channel width and mean fluid velocity.
- The scaling factor of 13 is used to compare different channel widths.
- The lift curve shape is found to depend on $h/W$ and $R_{ch}$ for regions away from the wall.
- The magnitude of $R_G$ depends on $W/d$.

**References:**

Fig. 19. Comparison of lift curves in channels of different widths at given values of channel Reynolds numbers. (a) \( R_\text{ch} = 60 \); (b) \( R_\text{ch} = 300 \); (c) \( R_\text{ch} = 480 \); (d) \( R_\text{ch} = 1200 \); (e) \( R_\text{ch} = 1920 \); (f) \( R_\text{ch} = 4800 \).

The above observation is useful to obtain a stability diagram, similar to that in Fig. 15, that includes the effect of the channel width. It would therefore be possible to predict e.g. particle-depleted zones in any channel.

Following the procedure used to obtain Fig. 15, we plot in Fig. 20a the boundary between the unstable and stable regions (instead of shading the unstable region as in Fig. 15) in \( h/W \) and \( R \) parameter space for different channel widths. Note that the line near the walls for \( W/d = 6 \) is incomplete because we could not perform simulations with the particle too close to the wall. As noted before, the boundary between the stable and unstable regions imply the locations of maxima or minima in the plot of \( R_G \) vs. the height of the particle (ref. Fig. 8). Thus Fig. 20a de-
pends on the shape of the lift curve. It is to be expected that the plot for different channel widths, in Fig. 20a, do not fall on top of one another since the shape of the lift curve does not scale with $R$ (as discussed above). Following the observation in Fig. 19, above, we plot in Fig. 20b the boundary between the unstable and stable regions in $h/W$ and $R_{ch}$ (instead of $R$) parameter space for
different channel widths. The plots for different channel widths overlap. There is some discrepancy in the region next to the walls where, as remarked before, neither the shape of the lift curve nor its magnitude scale with $h/W$ and $Re_b$.

Fig. 20b helps to identify regions of unstable equilibrium in channels of different widths unlike the result in Fig. 15 which was restricted to $W/d = 12$. The unstable equilibrium regions will be the particle-depleted zones in Poiseuille flows of dilute suspensions.

Remarks regarding comparison with the three-dimensional case are in order. Asmolov [1] has presented analytic results for 3D Poiseuille flow and compared them with the experiments of Segré and Silberberg [15]. The following observations are relevant to the results presented here:

1. Comparison of Figs. 8 and 13b from Asmolov [1], for the lift force, with e.g. Fig. 7 in this paper show qualitative similarity away from the wall region. However, near the wall region, their plots do not show the turning points that are observed at higher Reynolds numbers in this work (Fig. 7). This could be due to the assumptions involved in their theory.
2. Although the lift plots of Asmolov [1] and this work are in qualitative agreement near the channel centerline, they apparently do not agree with the experimental values deduced from the data of Segré and Silberberg [15]—see Fig. 13b of Asmolov [1]. One possible reason suggested is that Asmolov [1] has channel geometry while the experiments were in a pipe geometry.
3. The equilibrium positions for a neutrally buoyant particle are much closer to the walls in case of spheres compared to the cylinders in this work. This also reflects on the change in the equilibrium location as the channel Reynolds number increases. In this work the equilibrium position shifts towards the channel centerline while in the 3D case it shifts towards the channel walls as the channel Reynolds number is increased.

Clearly, three-dimensional simulations and experiments, e.g. to differentiate the behavior in channel and pipe geometries, will shed more light on these observations. The 2D results presented here, however, form a baseline study and can guide the investigation of the computationally more intensive 3D problem.

5. Multiple equilibrium positions in Oldroyd-B fluids

In Section 3 we considered the effect of fluid elasticity on lift-off i.e. on the lift force close to the wall. In this section we consider the effect of elasticity on the equilibrium diagram and the lift force in Oldroyd-B fluids. We perform constrained simulations (particle is fixed in the $y$-direction) for $E = 0.05, 0.1$ and $0.2$. Rest of the parameters are the same as those used in the simulations in Section 4.1. The ratio of retardation and relaxation times $\lambda_2/\lambda_1$ is 0.125 and $W/d = 12$.

The plot of $R_G$ as function of the height of the particle center at different values of shear Reynolds numbers for $E = 0.05$ is shown in Fig. 21. Plots for $E = 0.1$ and $0.2$ were similar. These plots imply multiple equilibrium positions even in Oldroyd-B fluids.

It is seen that the lift force on a particle at a given position and Reynolds number increases with the fluid elasticity. Increasing of the value of fluid elasticity makes the particle move toward the center of the channel.
Fig. 21. Plot of $R_G$ vs. $h/d$ at different values of $R$ for $E = 0.05$ and $\lambda_2/\lambda_1 = 0.125$. The bottom wall is $h/d = 0$ and the channel centerline is $h/d = 6$. Plot in the other half of the channel is anti-symmetric.

The effect mentioned above is better expressed in Fig. 22. The stable equilibrium position of a neutrally buoyant particle is closest to the channel centerline for the largest value of $E$. A comparison of the equilibrium height in Newtonian and Oldroyd-B fluids for a particle with $R_G = 9.81$ is

Fig. 22. The equilibrium diagram for a neutrally buoyant particle ($R_G = 0$) at $E = 0.0$ and $0.1$. The unstable equilibrium position for all cases is on the centerline.
shown in Fig. 23. The turning point is also observed in the viscoelastic case. The fluid elasticity shifts up the equilibrium position for the same Reynolds number and shifts the turning points towards lower Reynolds numbers.

Fig. 24 (similar to Fig. 15) identifies the unstable equilibrium regions in the channel for the Newtonian and Oldroyd-B cases. Fig. 24 shows that the fluid elasticity does not significantly alter
the unstable equilibrium region near the centerline for the values we have considered. It does affect the unstable regions near the wall. The fluid elasticity shifts this unstable region to higher Reynolds numbers. Once again, we note that Fig. 24 predicts that even in viscoelastic fluids there can be regions to which no particles will migrate irrespective of their weight in Poiseuille flows of dilute suspensions.

6. Conclusions

We present a brief summary of the results in this paper:

1. A power law correlation for the critical condition for lift-off in Oldroyd-B fluids is obtained assuming a surface roughness (or the particle separation from the wall) of 0.001d. It is seen that the fluid elasticity reduces the critical shear Reynolds number for lift-off. The lift force on the particle is seen to be singular as the gap size approaches zero; in qualitative agreement with the previous theoretical predictions. Power laws, with respect to the Reynolds number, are at the foundation of the correlation for lift. The effect of elasticity is found to be linear.

2. Multiple equilibrium positions of a particle in Poiseuille flows of Newtonian fluids are studied in detail. Different types of equilibrium diagrams are obtained. In particular a diagram for the equilibrium height of a neutrally buoyant particle is presented. The equilibrium height of the particle is seen to depend on the Reynolds number. The particle attains an equilibrium position closer to the channel centerline as the Reynolds number is increased.

3. It is shown that a heavy particle can also have a stable equilibrium position in the top half of the channel. This may appear counter-intuitive where one might expect the heavy particle to stay in the bottom half of the channel due to its weight.

4. We observe that, at a given Reynolds number, there are regions inside the channel in which no particles, irrespective of their density, can attain a stable equilibrium position. This would result in particle-depleted zones in a channel with Poiseuille flows of a dilute suspension of particles of varying densities. A diagram (Fig. 22) is presented to identify the particle-depleted zones in channels of different widths.

5. It is shown that the turning points in the plot of lift force are closely described by power law correlations. These correlations lead to the prediction of the range of Reynolds numbers or pressure gradients that will give either two equilibrium positions or one equilibrium position for a given particle.

6. It is identified that the shape of lift vs. particle height curve scales with the channel Reynolds number (i.e. independent of the particle size) in the regions away from the wall. Its magnitude does depend on the particle size. In the regions close to the wall, both, the shape and the magnitude of lift curve depend on the particle size.

7. We report multiple equilibrium positions for particles in Oldroyd-B fluids. This is due to two turning points in the equilibrium solution. The location of the turning point is shifted toward lower Reynolds numbers due to the fluid elasticity.

8. The equilibrium position of a particle at a given value of density and pressure gradient is closer to the channel centerline as the fluid elasticity increases.
9. Particle-depleted zones are also identified in the Oldroyd-B fluid case. The range of the unstable region near the channel centerline is not significantly changed at the elasticity numbers we considered. The two unstable regions near the walls are shifted toward higher Reynolds numbers as the fluid elasticity is increased.

Acknowledgments

This work was partially supported by the National Science Foundation KDI/New Computational Challenge grant (NSF/CTS-98-73236) and by the Minnesota Supercomputer Institute. NAP acknowledges Northwestern University for its support through the startup funds.

References