Kolmogorov Two-Thirds Law by Matched Asymptotic Expansion
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Abstract. The Kolmogorov two-thirds law is derived for large Reynolds number isotropic
turbulence by the method of matched asymptotic expansions. Inner and outer variables
are derived from the Karman-Howarth equation by using the von Karman self-preservation
hypothesis. Matching the resulting large Reynolds number asymptotic expansions yields
the Kolmogorov law. The Kolmogorov similarity hypotheses are not assumed; only the
Navier-Stokes equation is employed and the assumption that dissipation is finite. This
indicates that the Kolmogorov results are a direct consequence of the Navier-Stokes equa-
tions.

I. INTRODUCTION.
The Karman-Howarth equation\textsuperscript{1} describes isotropic decaying turbulence in terms of
correlation moments. A slightly different form, derived by Landau and Lifshitz\textsuperscript{2}, in terms of
related structure functions, is

\[
-\frac{2}{3}\epsilon - \frac{1}{2} \frac{\partial B_{ll}}{\partial t} = \frac{1}{6r^4} \frac{\partial^4 B_{lll}}{\partial r^4} - \frac{\nu}{r^4} \frac{\partial}{\partial r} \frac{\partial^4 B_{ll}}{\partial r^4},
\]

where \(B_{ll}\) and \(B_{lll}\) are the second and third order longitudinal structure functions defined by

\[
B_{ll}(r,t) = \left\langle (\hat{r} \cdot \mathbf{v}(\mathbf{x}_1 + r\hat{r}, t) - \hat{r} \cdot \mathbf{v}(\mathbf{x}_1, t))^2 \right\rangle
\]

\[
B_{lll}(r,t) = \left\langle (\hat{r} \cdot \mathbf{v}(\mathbf{x}_1 + r\hat{r}, t) - \hat{r} \cdot \mathbf{v}(\mathbf{x}_1, t))^3 \right\rangle
\]

\(r\) being the separation between two points in isotropic turbulent flow along the direction
defined by \(\hat{r}\). The dissipation \(\epsilon\) is defined by

\[
\epsilon(t) = -\frac{3}{2} \frac{dU^2}{dt} = \left. \frac{15\nu}{2} \frac{d^2 B_{ll}}{dr^2} \right|_0
\]

with

\[
U(t)^2 = \left\langle (\hat{r} \cdot \mathbf{v}(\mathbf{x}_1, t))^2 \right\rangle
\]

the “energy” in one component of the velocity field. Assuming that turbulent velocities are
uncorrelated at infinite separation, \(B_{ll} \to 2U^2\) as \(r \to \infty\). Equation\textsuperscript{(4)} provides a boundary
condition as \(r \to 0\). An outer length scale is defined by \(L = U^3/\epsilon\) and a Reynolds number
by \(R_L = UL/\nu\).

It should be emphasized that eq.\textsuperscript{(1)} is not a closed equation because \(B_{lll}\) is not known
a known functional of \(B_{ll}\). Recently Oberlack and Peters\textsuperscript{3} closed eq.\textsuperscript{(1)} with an eddy
viscosity model which they solved numerically. There have been numerous proposals for
closing the spectral form of eq.\textsuperscript{(1)} (see Monin and Yaglom\textsuperscript{4}). The objective of the present
paper is to see what can be obtained without a specific closure.
Von Karman\(^1\) sought self similar solutions of eq.(1) as if it were a closed partial differential equation. As such, eq.(1) describes a singular perturbation problem with a small parameter multiplying the term with the highest spatial derivative. The only physical parameters in the problem are those that appear in eq.(1) and its boundary conditions; \(U, \epsilon, \nu\). It is shown below that in addition to von Karman's similarity scaling (outer variables) there is a second similarity form which has Kolmogorov\(^5\) scaling (inner variables), describing a boundary layer at small separation, which removes the singularity. The resulting two similarity "solutions" are matched to get the Kolmogorov two-thirds law.

II. GENERAL SIMILARITY VARIABLES

The Karman-Howarth equation may be written in dimensionless form with a general change of variables, in preparation for a similarity analysis. With \(v(t)\) a velocity, and \(l(t)\) a length, one may write

\[
B_{il} = v(t)^2 b_{il}(x,t) \tag{6}
\]

\[
B_{iii} = v(t)^3 b_{iii}(x,t) \tag{7}
\]

where \(x = r/l(t)\). Equations(6) and (7) are the general form of the von Karman\(^1\) self-preservation hypothesis. (The paper specifically credits it to von Karman and not to both authors.) Noting that \(\partial x/\partial t = -x \dot{l}/l\), eq.(1) may be written

\[
-\frac{2}{3} P_1 - \frac{1}{2} \left( \frac{l}{v} \frac{\partial b_{il}}{\partial t} + P_2 b_{il} - P_3 x \frac{\partial b_{il}}{\partial x} \right) = \frac{1}{6x^4} \frac{\partial x^4 b_{iii}}{\partial x} - P_4 \frac{1}{x^4} \frac{\partial}{\partial x} x^4 \frac{\partial b_{il}}{\partial x}. \tag{8}
\]

The four dimensionless parameters which occur here are

\[
P_1 = \frac{\epsilon l}{v^3}, \quad P_2 = \frac{2vl}{v^2}, \quad P_3 = \frac{l}{v}, \quad P_4 = \frac{\nu}{vl}, \tag{9}
\]

where the overdot denotes a time derivative. There can be a similarity solution, with no explicit time dependence (the \(\partial b_{il}/\partial t\) term in eq.(8) can be dropped), if these parameters are either constant or become zero as \(R_L \to \infty\).

III. OUTER EXPANSION

Following von Karman, outer variables are taken to be

\[
v = U(t), \quad l = L(t). \tag{10}
\]

Then \(P_1 = 1, \ P_2 = -2/3\), and assuming

\[
U^2 = Ct^{-n}, \quad L = Dt^m, \tag{11}
\]

so that \(\epsilon = (3/2)n Ct^{-n-1}\), it is found that

\[
P_3 = \frac{L}{U} = \frac{Dm}{C^{1/2}} t^{m-1+n/2}. \tag{13}
\]
This will be constant if \( m = 1 - n/2 \). Since \( L = U^3/\varepsilon = (2/3n)C^{1/2} t^{1-n/2} \), and comparing with eq. (11), it is seen that \( D/C^{1/2} = 2/3n \), and therefore

\[
P_3 = \frac{2}{3n} \left( 1 - \frac{n}{2} \right).
\]

(14)

Since \( P_4 = R_L^{-1} \) similarity can be achieved for infinite Reynolds number.

The energy decay exponent \( n \) will be considered to be a known parameter. It has been shown by Gad-el-hak and Corrsin\(^6\) that it depends on grid parameters for active-grid generated turbulence, suggesting that it depends on initial conditions. The general theoretical situation for \( n \) is deferred to the Appendix, where a generalized Saffman\(^7\) invariant is proposed.

Inserting the parameter values into eq. (8), but not yet assuming similarity, it may be written

\[
-\frac{2}{3} - \frac{1}{2} \left( \frac{2}{3n} t \frac{\partial b_{li}}{\partial t} - \frac{2}{3} b_{li} - \frac{2}{3n} \left( 1 - \frac{n}{2} \right) x \frac{\partial b_{li}}{\partial x} \right) = \frac{1}{6x^4} \frac{\partial x^4 b_{li}}{\partial x} - \frac{1}{R_L} \frac{\partial}{\partial x} x^4 \frac{\partial b_{li}}{\partial x}.
\]

(15)

The small parameter multiplying the highest derivative indicates that this is a singular perturbation problem. This equation has an outer asymptotic expansion of form

\[
b_{li}(x, t) = b_{li}^{01}(x) + R_L^{-1} b_{li}^{02}(x) + \ldots
\]

(16)

where \( R_L \) is large (and time dependent), and \( x = x_o = r/L \). This is of the form of a similarity solution plus a small time dependent correction. The lowest order terms satisfy

\[
-\frac{2}{3} + \frac{1}{3} \left( b_{li}^{01} + \frac{1}{n} \left( 1 - \frac{n}{2} \right) x \frac{\partial b_{li}^{01}}{\partial x} \right) = \frac{1}{6x^4} \frac{\partial x^4 b_{li}^{01}}{\partial x}.
\]

(17)

**IV. INNER EXPANSION**

It is desired to rescale in such a way that the viscous term is retained. Therefore take \( P_4 = \nu/\nu_l = 1 \) and \( P_1 = \varepsilon \nu/\nu^3 = 1 \). These two equations give

\[
l = LR_L^{-3/4}, \quad v = UR_L^{-1/4}.
\]

(18)

It is not too surprising that \( l \equiv (\nu^3/\varepsilon)^{1/4} \) and \( v \equiv (\nu\varepsilon)^{1/4} \) are the Kolmogorov\(^5\) scales. The other two parameters are \( P_2 = a_1 R_L^{-1/2} \) and \( P_3 = a_2 R_L^{-1/2} \), where \( a_1 = -(1 + n)/3n \) and \( a_2 = (1 + n)/6n \) are constants. Clearly \( P_2 \) and \( P_3 \) will tend to zero as \( R_L \to \infty \), and a similarity solution is again possible in the limit.

In this scaling eq. (8) takes the form

\[
-\frac{2}{3} - \frac{1}{2} \left( \frac{2}{3n} t \frac{\partial b_{li}}{\partial t} + a_1 b_{li} - a_2 x \frac{\partial b_{li}}{\partial x} \right) R_L^{-1/2} = \frac{1}{6x^4} \frac{\partial x^4 b_{li}}{\partial x} - \frac{1}{x^4} \frac{\partial}{\partial x} x^4 \frac{\partial b_{li}}{\partial x}
\]

(19)

where here \( x = x_i = (r/L) R_L^{3/4} \). This equation has an expansion of form

\[
b_{li}(x_i, t) = b_{li}^{i1}(x_i) + R_L^{-1/2} b_{li}^{i2}(x_i) + \ldots
\]

(20)
That is, it is another similarity solution plus a small time dependent correction. The lowest order terms satisfy

\[
\frac{2}{3} = \frac{1}{6x^4} \frac{\partial x^4 b_i^{i,1}}{\partial x} - \frac{1}{x^4} \frac{\partial}{\partial x} x^4 \frac{\partial b_i^{i,1}}{\partial x}.
\]

(21)

V. MATCHED ASYMPTOTIC EXPANSIONS

When there are two different small parameter asymptotic expansions of a function, with its variables scaled in different ways by the small parameter, the method of matched asymptotic expansions may be employed. This subject, developed in books by Cole\(^8\) and Van Dyke\(^9\) says basically that the inner expansion of the outer expansion should equal the outer expansion of the inner expansion. The form used by Van Dyke is a little easier to use: the M-term inner expansion of the N-term outer expansion is equal to the N-term outer expansion of the M-term inner expansion. This is applied below with M and N equal one.

A one term inner expansion is

\[
B_{iI}^i = U^2 R_L^{-1/2} b_i^{i,1}(x_i) \equiv U^2 R_L^{-1/2} b_i^{i,1}(x_o R_L^{3/4})
\]

(22)

expressing it in outer variables. Note that the inner and outer variables are related by \(x_i = x_o R_L^{3/4}\). A one term outer expansion is

\[
B_{iI}^0 = U^2 b_i^{o,1}(x_o).
\]

(23)

The required operations proceed as follows. The outer expansion of eq.(22) is obtained from the limiting process \(R_L \rightarrow \infty\) holding \(x_o\) constant. This is the same as the asymptotic expansion of the function for large values of its argument. Denote this function by an overtilde;

\[
U^2 R_L^{-1/2} \tilde{b}_i^{i,1}(x_o R_L^{3/4}).
\]

(24)

The inner expansion of eq.(23) is obtained by expressing the function in inner variables,

\[
U^2 b_i^{o,1}(x_i R_L^{-3/4})
\]

(25)

and expanding as \(R_L \rightarrow \infty\) holding \(x_i\) fixed. This is the same as the asymptotic expansion of this function for small values of its argument. Denote this by

\[
U^2 \tilde{b}_i^{o,1}(x_o),
\]

(26)

expressing it back in outer variables for comparison.

Equating (24) and (26)

\[
U^2 R_L^{-1/2} \tilde{b}_i^{i,1}(x_o R_L^{3/4}) = U^2 \tilde{b}_i^{o,1}(x_o).
\]

(27)

Since the right hand side doesn’t depend on \(R_L\) and the left hand side depends on it, the conclusion to be drawn is that this functional equation must have solution
\[ \tilde{b}^{i,1}_{ll}(x_i) = C_K x_i^{2/3}, \quad (28) \]
\[ \tilde{b}^{o,1}_{ll}(x_o) = C_K x_o^{2/3} \quad (29) \]
in order to have a finite limit as \( R_L \to \infty \). This makes eq.(27) an identity. Therefore the inner expansion goes like \( r^{2/3} \) for large argument while the outer expansion goes like \( r^{2/3} \) for small argument. In terms of \( B_{ll} \) the overlapping part of the expansions is
\[ B_{ll} = C_K U^2 \left( \frac{T}{L} \right)^{2/3} \equiv C_K (er)^{2/3} \quad (30) \]
which is the Kolmogorov result. The Kolmogorov prefactor \( C_K \) is not determined. Experimentally it is about 2.

This analysis may be carried further to obtain the Kolmogorov “4/5” law. Equation (17) may be integrated to get \( b^{o,1}_{ll} \) in terms of \( b^{i,1}_{ll} \);
\[ b^{o,1}_{ll}(x_o) = -\frac{4}{5} x_o + \frac{2}{x_o^4} \int_0^{x_o} x^4 \left( b^{i,1}_{ll}(x) + \frac{1}{n} \left( 1 - \frac{n}{2} \right) x \frac{\partial b^{o,1}_{ll}}{\partial x} \right) dx . \quad (31) \]
Substituting eq.(29) for \( b^{o,1}_{ll} \) gives
\[ b^{o,1}_{ll}(x_o) = -\frac{4}{5} x_o + \frac{4C_K}{17} \frac{1 + n}{n} x_o^{5/3} . \quad (32) \]

Therefore
\[ B_{lll}^{o,1}(r,t) = U^3 \left( -\frac{4}{5} x_o + \frac{4C_K}{17} \frac{1 + n}{n} x_o^{5/3} + \ldots \right) \quad (33) \]
as \( x_o \to 0 \). The second term becomes smaller than the first for small \( x_o \).

In the same way eq.(21) may be integrated to get
\[ b^{i,1}_{ll} = -\frac{4}{5} x_i + 6 \frac{\partial b^{i,1}_{ll}}{\partial x_i} , \quad (34) \]
and substituting eq.(28) for \( b^{i,1}_{ll} \) yields
\[ b^{i,1}_{ll} = -\frac{4}{5} x_i + 4C_K x_i^{-1/3} \quad (35) \]
and therefore
\[ B_{lli}^{i,1}(r,t) = U^3 R_L^{-3/4} \left( -\frac{4}{5} x_i + 4C_K x_i^{-1/3} + \ldots \right) \quad (36) \]
as \( x_i \to 0 \). Here the second term becomes smaller for large \( x_i \). Matching eqs.(33) and (36) in the usual way (inner of outer equals outer of inner) retains only the “4/5” parts;
\[ B_{lll} = -\frac{4}{5} U^3 \frac{r}{L} \equiv -\frac{4}{5} er , \quad (37) \]
which is the Kolmogorov “4/5” law.

Because of the additional terms in eqs.(33) and (36) it is possible to estimate the overlap for the “4/5” law, which could be called the extent of the “inertial subrange”. Choosing 10% discrepancy ranges by

\[
\frac{4C_K}{17} \frac{1 + n}{n} x_o^{5/3} < 0.1 \frac{4}{5} x_o
\]  

(38)

from eq.(33), and

\[
4C_K x_i^{-1/3} < 0.1 \frac{4}{5} x_i
\]  

(39)

from eq.(36), it is seen that the deviation of \( B_{\text{ill}} \) from eq.(37) will be less than 10% if \( x_o \) is restricted to the range

\[
R_L^{-3/4} \left( \frac{5C_K}{1} \right)^{3/4} < x_o < \left( 0.1 \frac{17}{5C_K} \frac{n}{1 + n} \right)^{3/2}.
\]  

(40)

For definiteness let \( C_K = 2 \) and \( n = 1.2 \), which are close to accepted experimental values. Then eq.(40) can be simplified to

\[
32 R_L^{-3/4} < x_o < 0.28
\]  

(41)

or

\[
32 < x_i < 0.28 R_L^{3/4}.
\]  

(42)

For example, for \( R_\lambda = 2000 \) (using \( R_L = R_\lambda^2/15 \)) eq.(42) becomes

\[
32 < x_i < 328.
\]  

(43)

By this estimate there is only a single decade inertial subrange, even for this very high Reynolds number.

This result may be presented in a different way by constructing a composite expansion of form “the inner expansion plus the outer expansion minus the common part”. Adding eq.(33) and eq.(36) and subtracting eq.(37) gives

\[
B_{\text{ill}}^{\text{comp}} = U^3 \left( \frac{4C_K}{17} \frac{1 + n}{n} \right) x_o^{5/3} + U^3 R_L^{-3/4} \left( -\frac{4}{5} x_i + 4C_K x_i^{-1/3} \right).
\]  

(44)

Taking \( C_K = 2 \) and \( n = 1.2 \) this may be written, in inner variables,

\[
-b_{\text{ill}}^{\text{comp}} \equiv -\frac{B_{\text{ill}}^{\text{comp}}}{U^3 R_L^{-3/4}} = \frac{3.34}{R_\lambda} x_i^{5/3} + \frac{4}{5} x_i - 8 x_i^{-1/3}.
\]  

(45)

This is plotted in Fig.(1) for \( R_\lambda = 1000, 2000, 3000 \) and 4000, along with two straight lines. The solid line is \( 4x_i/5 \) and the dashed line is \( .9 \) times \( 4x_i/5 \). There is less than 10% deviation from the “4/5” law when the curves are between the two lines. Since one can’t see this range very well these curves and lines have been replotted in “compensated” form
in Fig.(2), dividing by $4x_i/5$ so that the “4/5” law is the horizontal solid line. One can see that the $R_\lambda = 1000$ curve barely enters the 10 % region, and even the $R_\lambda = 4000$ curve does not have a very long inertial subrange. When plotted as in Fig.(1) the high Reynolds number curves seem to follow the “4/5” law fairly well, with the slopes approaching 4/5 as Reynolds number increases. The deviations from the “4/5” law are comparable to those found in measurements in the atmospheric boundary layer over the ocean (Van Atta and Chen). The highest Reynolds numbers in these experiments are of order $R_\lambda = 4000$ (estimated from the law of the wall for rough walls).

![Graph](image1)

Fig.1 Third order structure function versus $r/\eta$.

![Graph](image2)

Fig.2 Compensated third order structure function versus $r/\eta$.

VI. DISCUSSION

The Kolmogorov $r^{2/3}$ and “4/5” laws have been derived from the Karman-Howarth equation by the method of matched asymptotic expansions in the limit of infinite Reynolds number. It is implied that there are corrections to these results for large but finite Reynolds number. A new result in this paper is a Reynolds number correction to the “4/5” law obtained by getting more information from the Karman-Howarth equation by asymptotic matching methods. The most striking aspect of this is how slowly the “4/5” law is approached as Reynolds number tends to infinity. A straight line tangent to eq.(45) has coefficient $4/5 - 8.45R_\lambda^{-2/3}$ and the curvature at the tangency point is $-3.43R_\lambda^{-7/6}$. These results indicate the rate at which the “4/5” law is approached and are consistent with the figures. Mydlarski and Warhaft noted a similar slow approach to the Kolmogorov spectrum. From an empirical fit to grid turbulence measurements they found that the spectral exponent is $-5/3 + 5.23R_\lambda^{-2/3}$ with the same slow $R_\lambda^{-2/3}$ approach to the asymptote.

The following discussion points are added for general interest.

1. The Kolmogorov results have sometimes been questioned because they appear not to depend on the Navier-Stokes equation. The present approach does depend on the Navier-Stokes equations and the assumption that the dissipation is finite in the limit of infinite
Reynolds number. The latter assumption rules out similar conclusions for two-dimensional turbulence which has been shown to have zero dissipation in the limit of infinite Reynolds number (Batchelor\textsuperscript{11}).

2. The matched asymptotic derivation of the \( r^{2/3} \) law is similar to Millikan’s\textsuperscript{13} derivation of the logarithmic law of the wall in a turbulent boundary layer by matching two empirical similarity laws, the law of the wall (inner) and the velocity defect law (outer).

3. With some additional assumptions the results derived here may be obtained by dimensional analysis and matched expansions, without explicit use of similarity solutions of the Karman-Howarth equation. Assume that the structure functions depend only on \( r, U, \epsilon, \nu \). These are more parameters than are contained in the Kolmogorov hypotheses. Kolmogorov restricted to \( r \ll L \), here the the full range of \( r \) values is under consideration, hence the inclusion of \( U \) in the parameter set. On dimensional grounds there are two possible scalings: von Karman scaling \( U, L = U^3/\epsilon \) and Kolmogorov scaling \( v = (\nu^3/\epsilon)^{1/4} \equiv U R_L^{-1/4}, l = (\nu^3/\epsilon)^{1/4} \equiv R_L^{-3/4} \). The \( p \)th order structure function may be expressed in two different dimensionless forms

\[
B_p(r, U, \epsilon, \nu) = U^p b^{o_p}(r/L, R_L)
\]

and

\[
B_p(r, U, \epsilon, \nu) = v^p b^{i_p}(r/l, R_L)
\]

Then, assuming that \( b^{o_p} \) and \( b^{i_p} \) can be expanded in inverse fractional powers of \( R_L \),

\[
b^{o_p}(r/L, R_L) = b^{o,1}(r/L) + \ldots ,
\]

\[
b^{i_p}(r/l, R_L) = b^{i,1}(r/l) + \ldots .
\]

This last assumption is not as well motivated as when the Karman-Howarth structure was used since there is no indication of of the order of the fractional powers. Now, Van Dyke’s asymptotic matching principle gives the equivalent of eq.(27);

\[
U^p R_L^{-p/4} b^{i,1}_p \left( \frac{T}{L} R_L^{3p/4} \right) = U^p b^{o,1}_p \left( \frac{T}{L} \right) .
\]

In order to get a finite result as \( R_L \to \infty \), this yields

\[
b^{i,1}_p(x_i) = C_p x_i^{p/3}, \quad b^{o,1}_p(x_o) = C_p x_o^{p/3}
\]

and

\[
B_p = C_p U^p (r/L)^{p/3} \equiv C_p (\epsilon r)^{p/3} .
\]

This last result is ascribed to Kolmogorov.

**APPENDIX.** Generalized Saffman Invariant
A general theoretical result may be deduced from the Karman-Howarth equation. If the longitudinal correlation function, \( R_{ll} = U^2 - B_{ll}/2 \), tends to zero algebraically as \( r \to \infty \), so that \( B_{ll} = 2U^2 - Ar^{-\alpha} \) as \( r \to \infty \), then it follows easily from eq.(1) that

\[
dA/\ dt = 0 ,
\]

for any Reynolds number, assuming only that \( B_{ll} \to 0 \) faster than \( r^{1-\alpha} \). Equation(53) was observed by Oberlack and Peters\(^3\) for similarity solutions of the eddy viscosity model, but it has more general significance. It implies that \( A \) is a dimensional invariant, a generalized Saffman invariant, which persists from the initial conditions and therefore can provide an additional condition to complete the similarity analysis. \( A \) has the same dimensions as \( U^2 L^\alpha \). Setting this to a constant, and using eq.(11), gives

\[
n = \frac{2\alpha}{2 + \alpha} .
\]

The Saffman\(^7\) invariant,

\[
I_S = \int_0^\infty r^2 R_{ll} dr \equiv \lim_{r \to \infty} (r^3 R_{ll}) ,
\]

is a special case with \( \alpha = 3 \), giving \( n = 1.2 \). Saffman\(^7\) showed, from the Karman-Howarth equation, that \( I_S \) is independent of time and that it has special physical significance for the persistence of large eddies with “impulse”. The Loitsiansky invariant (see ref.4) is

\[
I_L = \int_0^\infty r^4 R_{ll} dr ,
\]

requiring \( \alpha > 5 \) for the integral to exist. It gives the similarity condition \( U^2 L^5 = \text{const.} \), which with eq.(11), leads to \( n = 10/7 \). This is clearly at odds with eq.(54) which gives a larger value of \( n \) when \( \alpha > 5 \). An explanation for this is that \( A \equiv 0 \) for any \( \alpha > 5 \), implying that \( R_{ll} \) decays faster that algebraically whenever the Loitsiansky invariant exists, negating the original algebraic decay assumption. In partial support of this conclusion Oberlack and Peters showed numerically, with the eddy viscosity model, that \( n \) is given by eq.(54) for \( 2 < \alpha < 5 \), implying that \( 1 < n < 10/7 \), while for \( \alpha > 5 \) they found \( n = 10/7 \).

The arguments given above can be strengthened, independently of any similarity considerations, by considering the energy spectrum. By a technical estimate, whenever the Loitsiansky invariant exists, the energy spectrum function has the property \( E(k) = I_L k^4/3\pi + \) a remainder which tends to zero faster than \( k^4 \) as \( k \to 0 \). Well known asymptotics of Fourier integrals give \( R_{ll} = Ar^{-\alpha} \) as \( r \to \infty \) whenever \( E(k) = Bk^{\alpha-1} \) as \( k \to 0 \) with \( B \) proportional to \( A \). Therefore \( B \) is an invariant equivalent to \( A \). When the spectral exponent is \( \leq 4 \), \( \alpha \leq 5 \), and the Loitsiansky integral cannot exist. There is no possibility that the spectral exponent is greater than 4, since \( I_L \) either exists or it doesn’t. Since \( \alpha > 5 \) would require that the spectral exponent be greater than 4, it follows that \( \alpha \) can never be greater than 5, and \( R_{ll} \) must have exponential decay whenever \( I_L \) exists.
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