Kolmogorov Turbulence by Matched Asymptotic Expansions

by

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The Kolmogorov\(^1\) (K41) inertial range theory is derived from first principles by analysis of the Navier-Stokes equation using the method of matched asymptotic expansions without assuming isotropy or homogeneity and the Kolmogorov\(^2\) (K62) refined theory is analysed. This paper is an extension of Lundgren\(^3\), in which the second and third order structure functions were determined from the isotropic Karman-Howarth\(^4\) equation. The starting point for the present analysis is an equation for the difference in velocity between two points, one of which is a Lagrangian fluid point and the second, slaved to the first by a fixed separation \(r\), is not Lagrangian. The velocity difference, so defined, satisfies the Navier-Stokes equation with spatial variable \(r\). The analysis is carried out in two parts. In the first part the physical hypothesis is made that the mean dissipation is independent of viscosity as viscosity tends to zero, as assumed in K41. This means that the mean dissipation is finite as Reynolds number tends to infinity and leads to the K41 inertial range results. In the second part this dissipation assumption is relaxed in an attempt to duplicate the K62 theory. While the K62 structure is obtained, there are restrictions, resulting from the analysis which show that there can be no inertial range intermittency as Reynolds number tends to infinity, and therefore the mean dissipation has to be finite as Reynolds number tends to infinity, as assumed in part one. Reynolds number dependent corrections to the K41 results are obtained in the form of compensating functions of \(r/\lambda\), which tend to zero slowly like \(R_\lambda^{-2/3}\) as \(R_\lambda \rightarrow \infty\).
I. INTRODUCTION

The purpose of this paper is to study the statistics of the velocity difference between two neighboring points in a turbulent flow in order to derive the inertial range two-thirds law of Kolmogorov\textsuperscript{1}. Kolmogorov obtained this celebrated result by dimensional analysis and considerable physical insight. It is shown here that it follows from first principles by analysis of the Navier-Stokes equation.

It is natural to try to derive an equation for the velocity difference by subtracting two Navier-Stokes equations at fixed points. One finds an equation which has the time derivative of the velocity difference, but it is not useful because the rest of the equation depends on both velocities separately and not merely on their difference. It is shown below that if one of the points is a Lagrangian particle, the resulting equation depends only on the velocity difference and is susceptible to further analysis. A similar equation is derived in Monin and Yaglom\textsuperscript{5}, p.401.

Let one of the points be a Lagrangian fluid particle represented by \( \mathbf{x} = \mathbf{x}(\mathbf{X}, t) \), where \( \mathbf{X} \) is a Lagrangian variable. The second point at \( \mathbf{x} + \mathbf{r} \) is not Lagrangian. The velocity difference is defined by

\[
\mathbf{v}(\mathbf{r}, t; \mathbf{X}) \equiv \mathbf{u}(\mathbf{x}(\mathbf{X}, t) + \mathbf{r}, t) - \mathbf{u}(\mathbf{x}(\mathbf{X}, t), t) \tag{1}
\]

The velocity field \( \mathbf{u}(\mathbf{x}, t) \) is a realization of a turbulent incompressible flow; the vector \( \mathbf{r} \) is an independent spatial variable. The usual Lagrangian kinematic notation and definitions\textsuperscript{6,7} are employed. The transformation

\[
\mathbf{x} = \mathbf{x}(\mathbf{X}, t) \tag{2}
\]

gives the position at time \( t \) of the fluid particle which was at \( \mathbf{X} \) at an initial time. The velocity field at point \( \mathbf{x} \) is defined in terms of this transformation by

\[
\mathbf{u}(\mathbf{x}, t) = \left. \frac{\partial \mathbf{x}(\mathbf{X}, t)}{\partial t} \right|_{\mathbf{x}} = \frac{\partial \mathbf{x}}{\partial t}, \tag{3}
\]

i.e. the velocity at a point \( \mathbf{x} \) at time \( t \) is defined as the velocity of a Lagrangian particle which passes through \( \mathbf{x} \) at this time. The inverse transformation

\[
\mathbf{X} = \mathbf{X}(\mathbf{x}, t) \tag{4}
\]

gives the initial position of the particle which is at \( \mathbf{x} \) at time \( t \).

Differentiating (1) with respect to \( t \), holding both \( \mathbf{X} \) and \( \mathbf{r} \) fixed and using the Navier-Stokes equation gives

\[
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial \mathbf{r}} = -\frac{1}{\rho} \frac{\partial p'}{\partial \mathbf{r}} + \nu \frac{\partial}{\partial \mathbf{r}} \cdot \nabla \mathbf{v} - \frac{\partial \mathbf{u}}{\partial t} \tag{5}
\]

where

\[
p' = p(\mathbf{x}(\mathbf{X}, t) + \mathbf{r}, t) - p(\mathbf{x}(\mathbf{X}, t), t) \tag{6}
\]

and

\[
\frac{d \mathbf{u}}{dt} = \left. \frac{\partial \mathbf{u}(\mathbf{x}(\mathbf{X}, t), t)}{\partial t} \right|_{\mathbf{x}} \tag{7}
\]
is the acceleration of the material point, a function of time alone since \( X \) is fixed. Equation (5) will be recognized as the Navier-Stokes equation in an accelerated coordinate frame, hence the acceleration \( du/dt \) on the right hand side - an inertial force. Note that all the spatial derivatives are with respect to \( r \), none are with respect to \( X \). Since \( du/dt \) is a function of time alone it can be absorbed into the pressure as a time dependent hydrostatic pressure; it has no dynamical effect for an incompressible flow. Equation (5) may thus be rewritten

\[
\frac{\partial v}{\partial t} + v \cdot \frac{\partial v}{\partial r} = -\frac{1}{\rho} \frac{\partial P}{\partial r} + \nu \frac{\partial}{\partial r} \cdot \frac{\partial}{\partial r} v
\]  

(8)

with

\[
P = p' + \rho v \cdot \frac{du}{dt} .
\]

(9)

Because \( u(x, t) \) is divergence free it is easily seen that

\[
\frac{\partial}{\partial r} \cdot v(r, t; X) = 0 ,
\]

(10)

giving the additional condition required to determine \( P \). If \( u(r, 0) = u_0(x) \) is given statistically at some initial time, then because of the identity \( x(X, 0) \equiv X \), the initial condition for \( v \) is \( v(r, 0; X) = u_0(X + r) - u_0(X) \). Therefore the velocity difference satisfies the ordinary Navier-Stokes equation with the obvious additional condition that the origin, \( r = 0 \), has to be a stagnation point.

It is desired to relate solutions of (8) and (10) to the statistics of the turbulent velocity field at fixed points in space \( x + r \) and \( x \). This is done conceptually by using the inverse transformation (4), writing the velocity difference as

\[
u(x + r, t) - u(x, t) = v(r, t; X(x, t)) .
\]

(11)

That is, one looks at the velocity difference as the Lagrangian particle passes through the position \( x \). Ensemble averages at a fixed \( x \) are performed by collecting an ensemble of solutions with differing values of \( X \) which have the same \( x \) at time \( t \). Statistics based on such an ensemble are the same as if both points were fixed in space. Dependence on \( X \), which can occur only as a parameter, should be regarded as dependence on a random variable.

II. ANALYSIS YIELDING K41 RESULTS

The K41 related analysis, with the assumption that dissipation is finite as Reynolds number tends to infinity, will be presented first because it is complete. This will set the stage for an analysis of the K62 hypothesis which will be presented in III and will be shown to justify the dissipation assumption.

The equations will be made dimensionless by using statistical parameters of the turbulent flow at a fixed point \( x \) at a fixed time \( t \). A characteristic velocity \( U \) may be defined from the turbulent energy by

\[
U^2 = \frac{1}{3} \left\langle (u - \langle u \rangle) \cdot (u - \langle u \rangle) \right\rangle
\]

(12)
and an outer length $L$ can be defined in terms of the homogeneous dissipation $\epsilon$ by

$$L = \frac{U^3}{\epsilon}$$  \hfill (13)

where the mean dissipation $\epsilon$ is defined by

$$\epsilon = \frac{1}{2} \nu \lim_{r \to 0} \frac{\partial}{\partial r} \cdot \frac{\partial}{\partial r} (\mathbf{v}(r, t; \mathbf{X}) \cdot \mathbf{v}(r, t; \mathbf{X}))$$ \hfill (14)

and the ensemble average is to be taken at fixed $\mathbf{x}$ by using $\mathbf{X}(\mathbf{x}, t)$ as described above. Homogeneous dissipation is the actual dissipation if the turbulence is homogeneous and in general it differs from the actual dissipation by terms which tend to zero as the Reynolds number tends to infinity. Homogeneity is not assumed here, but the limit as Reynolds number tends to infinity will be made. The assumption that $\epsilon$ is independent of viscosity in the limit ensures that $L$ is an appropriate outer length scale. This would not be the case if the dissipation failed to be finite in the limit. It is an accepted experimental fact that $L$ defined by (13) is of the same order as the size of energy containing eddies of the flow.

Using (8) and (10) as the starting point it is quite straightforward to derive the Kolmogorov results by the method of matched asymptotic expansions, generalizing the procedures initiated in reference 3.

A. Outer expansion

Define dimensionless outer variables by

$$\mathbf{v}_o = \frac{\mathbf{v}}{U}, \quad \mathbf{r}_o = \frac{\mathbf{r}}{L}, \quad P_o = \frac{P}{\rho U^2}, \quad \tau = \frac{tU}{L}.$$ \hfill (15)

Then (8) and (10) become

$$\frac{\partial \mathbf{v}_o}{\partial \tau} + \mathbf{v}_o \cdot \nabla \mathbf{v}_o = -\nabla P_o + R_L^{-1} \nabla^2 \mathbf{v}_o,$$ \hfill (16)

$$\nabla \cdot \mathbf{v}_o = 0$$ \hfill (17)

where $R_L = UL/\nu$ and $\nabla = \partial/\partial \mathbf{r}_o$. An outer expansion may be taken in the form

$$\mathbf{v} = U\mathbf{v}_o, \quad \mathbf{v}_o = \mathbf{v}_{o,1} + R_L^{-1} \mathbf{v}_{o,2} + \cdots.$$ \hfill (18)

The first term $\mathbf{v}_{o,1}$ satisfies (16) without the viscous term. The $R_L^{-1}$ term, which is motivated by the size of the neglected term, is not used in this analysis but will be shown to be of consistent order in subsection D.

B. Inner expansion

Change variables so that the viscous term will be retained as $R_L \to \infty$. Let inner variables be defined by

$$\mathbf{r}_i = R_L^{-\alpha} \mathbf{r}_o, \quad \mathbf{v}_i = R_L^{-\beta} \mathbf{v}_o, \quad P_i = R_L^{-2\beta} P_o.$$ \hfill (19)
Then (16) may be rewritten as

\[ R_L^{-\beta + \alpha} \frac{\partial v_i}{\partial \tau} + v_i \cdot \nabla v_i = -\nabla_i P_i + R_L^{-1 - \beta - \alpha} \nabla_i^2 v_i \ \text{(20)} \]

In order that both inertia and viscous terms be retained as \( R_L \to \infty \) it is necessary that

\[ \beta + \alpha = -1 \ \text{(21)} \]

An additional condition is required. Assume, with Kolmogorov\(^1\), that the dissipation is finite as Reynolds number tends to infinity. From (14) one finds

\[ \frac{\epsilon L}{U^3} = \frac{1}{2} R_L^{-1} \lim_{r \to 0} \nabla_i^2 \langle v_o \cdot v_o \rangle \]

\[ = \frac{1}{2} R_L^{-1 + 2\beta - 2\alpha} \lim_{r_i \to 0} \nabla_i^2 \langle v_i \cdot v_i \rangle \ \text{(22)} \]

In order for the right hand side to be finite as \( R_L \to \infty \) it is necessary that

\[ \beta - \alpha = \frac{1}{2} \ \text{(23)} \]

Equations (21) and (23) have solution

\[ \beta = -\frac{1}{4}, \ \alpha = -\frac{3}{4} \ \text{(24)} \]

therefore the inner and outer variables are related by

\[ r_i = R_L^{3/4} r_o, \ v_i = R_L^{1/4} v_o \ \text{(25)} \]

The effective inner length and velocity scales are Kolmogorov variables

\[ \eta = R_L^{-3/4} L = \left( \frac{v^3}{\epsilon} \right)^{1/4} \ \text{(26)} \]

and

\[ v_K = R_L^{-1/4} U = \left( v \epsilon \right)^{1/4}. \ \text{(27)} \]

With this choice of \( \alpha \) and \( \beta \) (20) becomes

\[ R_L^{-1/2} \frac{\partial v_i}{\partial \tau} + v_i \cdot \nabla v_i = -\nabla_i P_i + \nabla_i^2 v_i \ \text{(28)} \]

which suggests an inner expansion

\[ \mathbf{v} = U R_L^{-1/4} \mathbf{v}_i, \quad \mathbf{v}_i = \mathbf{v}_{i,1} + R_L^{-1/2} \mathbf{v}_{i,2} + \cdots \ \text{(29)} \]
C. Matching

Since the two asymptotic expansions describe the same function, the inner expansion of the outer expansion should equal the outer expansion of the inner expansion\(^8,9\), provided a region of overlap exists.

The inner expansion of the outer expansion is

\[
v = U(v_{o,1}(R_L^{-3/4}r_i) + R_L^{-1}v_{o,2}(R_L^{-3/4}r_i) + \cdots) \quad \text{as } R_L \to \infty, \quad r_i \text{ fixed}
\]

\[
\sim U v_{o,1}(r_o) \quad \text{as } r_o \to 0,
\]

keeping only one term.

The outer expansion of the inner expansion is

\[
v = UR_L^{-1/4}(v_{i,1}(R_L^{-3/4}r_o) + R_L^{-1/2}v_{i,2}(R_L^{-3/4}r_o) + \cdots) \quad \text{as } R_L \to \infty, \quad r_o \text{ fixed}
\]

\[
\sim UR_L^{-1/4}v_{i,1}(r_i) \quad \text{as } r_i \to \infty
\]

\[
= UR_L^{-1/4}v_{i,1}(R_L^{-3/4}r_o),
\]

expressing it in outer variables in (34) for matching purposes. Matching first terms (equating (31) and (34)) gives

\[
v_{o,1}(r_o) = R_L^{1/4}v_{i,1}(R_L^{3/4}r_o).
\]

This can be satisfied in the limit \(R_L \to \infty\) only if

\[
v_{o,1} = V r_o^{1/3}
\]

and

\[
v_{i,1} = V r_i^{1/3}.
\]

That is, substituting (36) and (37) into (35), with \(r_i = R_L^{-3/4}r_o\) satisfies it exactly and any power other than \(1/3\) would give zero or infinity as \(R_L \to \infty\). In (36) and (37) \(V = V(X, t, \hat{r})\), where \(\hat{r}\) is a unit vector in the direction of \(r\). The significant result is

\[
v = U V \left(\frac{r}{L}\right)^{1/3} = U R_L^{-1/4} V \left(\frac{r}{\eta}\right)^{1/3} = V (er)^{1/3}
\]

a 1/3 power law with a coefficient which is a random variable.

Structure function ensemble averages at a fixed point \(x\) are then given by

\[
\langle v \ldots p \text{ terms } \ldots v \rangle = \langle V \ldots p \text{ terms } \ldots V \rangle U p \left(\frac{r}{L}\right)^{p/3}
\]

\[
\equiv \langle V \ldots p \text{ terms } \ldots V \rangle (er)^{p/3}
\]

where \(X = X(x, t)\) in the \(V\) terms. If this were isotropic turbulence \(\langle V \ldots p \text{ terms } \ldots V \rangle\) would be an isotropic tensor function of \(\hat{r}\). Equation (40) is the general Kolmogorov\(^1\) inertial range structure function result. In the following, attention will be directed to longitudinal structure functions, defined by

\[
B_p = \langle (v \cdot \hat{r})^p \rangle.
\]

From (40)

\[
B_p = C_p (er)^{p/3}
\]

where \(C_p = \langle (V \cdot \hat{r})^p \rangle\) is a dimensionless constant.
D. Corrections to the K41 results

The first term in the outer expansion was found to satisfy

\[
\frac{\partial v_{o,1}}{\partial \tau} + v_{o,1} \cdot \nabla v_{o,1} = -\nabla o P_{o,1} , \quad \nabla o \cdot v_{o,1} = 0 . \tag{43}
\]

Matching determined

\[
v_{o,1} = V r_o^{1/3} \quad \text{as} \quad r_o \to 0 , \tag{44}
\]

with \( V \) depending on the direction \( \hat{r} \) but independent of \( r_o \) itself. It is desired to obtain the order of the next term in this small \( r_o \) expansion. Estimating the orders (in \( r_o \)) of the terms in (43) it is observed that (44) satisfies it, neglecting the time derivative which is of smaller order. A perturbation expansion which retains this next order is

\[
v_{o,1} = V r_o^{1/3} - V_2 r_o + \cdots \quad \text{as} \quad r_o \to 0 , \tag{45}
\]

with \( V_2 \) depending on \( \hat{r} \), but independent of \( r_o \); the sign of \( V_2 \) was chosen for future convenience. Expressing (45) in inner variables, in order to compute the two term inner expansion of the one term outer expansion, gives

\[
v = U \left( V(r_i R_L^{-3/4})^{1/3} - V_2(r_i R_L^{-3/4}) + \cdots \right) \tag{46}
\]

\[
= UR_L^{-1/4} \left( V r_i^{1/3} - R_L^{-1/2} V_2 r_i + \cdots \right) \tag{47}
\]

showing that the second term in the inner expansion is of order \( R_L^{-1/2} \), consistent with the second order term indicated in (29).

In a similar manner one can investigate the first term in the inner expansion, which satisfies

\[
v_{i,1} \cdot \nabla_i v_{i,1} = -\nabla_i P_{i,1} + \nabla_i^2 v_{i,1} , \quad \nabla_i \cdot v_{i,1} = 0 . \tag{48}
\]

By matching it was found that

\[
v_{i,1} = \bar{V} r_i^{1/3} \quad \text{as} \quad r_i \to \infty , \tag{49}
\]

the viscous term being of smaller order as \( r_i \to \infty \). A perturbation expansion which retains the viscous term to the next order, is seen to be

\[
v_{i,1} = \bar{V} r_i^{1/3} - \bar{W} r_i^{-1} + \cdots \quad \text{as} \quad r_i \to \infty . \tag{50}
\]

Consistency is checked by expressing this in outer variables and performing a two term outer expansion;

\[
v = UR_L^{-1/4} \left( V(r_o R_L^{3/4})^{1/3} - W(r_o R_L^{3/4})^{-1} + \cdots \right) \tag{51}
\]

\[
= U \left( \bar{V} r_o^{1/3} - R_L^{-1} \bar{W} r_o^{-1} + \cdots \right) . \tag{52}
\]

This is consistent with the indicated second order term in (18).
An additive composite expansion may be constructed by adding the inner and outer expansions, (50) and (45), and subtracting the common part (the first term in (45));

$$\frac{v_C}{U} = (V_r^{1/3} - V_2 r_o) + R_L^{-1/4} (V r_i^{1/3} - W r_i^{-1}) - V r_o^{1/3} .$$  \hspace{1cm} (53)

Restricting this to only the longitudinal components, and expressing it in inner variables yields

$$v_C \cdot \hat{r} = U R_L^{-1/4} V r_i^{1/3} (1 - \frac{V_2}{V} R_L^{-1/2} r_i^{2/3} - \frac{W}{V} r_i^{-4/3})$$  \hspace{1cm} (54)

where $V = v \cdot \hat{r}$, $V_2 = v_2 \cdot \hat{r}$ and $W = W \cdot \hat{r}$. This is the 1/3 power law times a compensating factor. This may be rewritten in a form which better brings out its essential features by using (40) and expressing the factor with $r/\lambda$ and $R_\lambda$ instead of $r_i$ and $R_L$. Using

$$r_i = (15)^{1/4} R_\lambda^{1/2} (r/\lambda) \hspace{1cm} R_L = R_\lambda^2 / 15$$  \hspace{1cm} (55)

it may be written

$$v_C \cdot \hat{r} = V (er)^{1/3} (1 - R_\lambda^{-2/3} f(r/\lambda))$$  \hspace{1cm} (56)

where

$$f(x) = A x^{2/3} + B x^{-4/3}$$  \hspace{1cm} (57)

and $A = (15)^{2/3} V_2 / V$ and $B = (15)^{-1/3} W / V$ are random variables. Assuming that $A$ and $B$ are positive the compensating factor has a maximum value at a position which scales with $\lambda$ and the 1/3 power law limit is approached slowly like $R_\lambda^{-2/3}$ as $R_\lambda \rightarrow \infty$.

Structure functions may be obtained by taking averages of powers of (56). Using (41),

$$B_p = (er)^{p/3} \langle V^p (1 - R_\lambda^{-2/3} f(r/\lambda))^p \rangle .$$  \hspace{1cm} (58)

If $p$ is not too large ($p R_\lambda^{-2/3} << 1$ is required) (58) may be expanded inside the expectation brackets, with the result

$$B_p = C_p (er)^{p/3} \langle 1 - R_\lambda^{-2/3} f_p(r/\lambda) \rangle$$  \hspace{1cm} (59)

with

$$f_p(x) = A_p x^{2/3} + B_p x^{-4/3}$$  \hspace{1cm} (60)

where $A_p = p < A V^p > / < V^p >$ and $B_p = p < B V^p > / < V^p >$. Equation (59) is the Kolmogorov$^1$ pth order structure function times a correction factor, usually called the compensated structure function, which depends on $R_\lambda$ and tends to unity as $R_\lambda$ tends to infinity.

From (59) one can see that the compensated structure functions of all orders are qualitatively similar. The functions are concave downwards, scale with the Taylor microscale $\lambda$ and flatten out slowly as $R_\lambda \rightarrow \infty$. These features can be seen explicitly for the compensated third order structure function which was derived in ref.3, and plotted in Fig.1 versus $r/\lambda$. These third order structure function results were also derived by Lindborg$^{10}$.
by a different analysis of the Karman-Howarth equation. In the figure \(-B_3/\epsilon r\) is plotted versus \(r/\lambda\) using (60) with \(C_3 = .8\) (the 4/5 law) and \(A_3 = 6.555, B_3 = 4.058\). These numbers come from ref.3, eq.\((45)\) or from ref.\(11, eq.(6)\) by a change of variable.

In Fig.2 the maximum value of \(-B_3/(\epsilon r)\) \((= .8 - 8.45 R_\lambda^{-2/3})\), which occurs at \(r/\lambda = 1.11\), is plotted versus \(R_\lambda\) along with a wide range of experimental values provided by Yves Gagne\(^1\) University of Grenoble). The data in grid turbulence \((R_\lambda = 44, 72, 110, 144)\), in a laboratory jet \((350, 453, 695)\) and in the huge return wind tunnel in ONERA-Modane \((2260, campaign of 1995)\) were reported in the Ph.D thesis of Yann Malecot \((1998, University of Grenoble)\). The higher \(R_\lambda\) data \((1700, 3600, 6100)\) were measured in the GReC experiment \((campaign of 2001)\) in a gaseous helium jet at 4° K. in CERN (Geneva). The agreement with the formula is quite outstanding. Figure 3 shows \(-B_3/\epsilon r\) data at \(R_\lambda = 350\) and 2260 versus \(r/\lambda\) compared with the theoretical results from (60). The agreement can be seen to be qualitatively correct even quite far from the maximum values.

The restriction that \(p\) not be too large in (59) can be dropped if it is assumed that \(A\) and \(B\) in (57) are constants. There is no justification for this but it is interesting to consider the consequences. Note, from (59) and (60) this makes \(A_p = pA, B_p = pB\) and since \(A_3\) and \(B_3\) are known one finds \(A = 2.185, B = 1.353\). With \(A\) and \(B\) constant the average in (58) may be written

\[
B_p = C_p (\epsilon r)^{p/3} (1 - R_\lambda^{-2/3} f(r/\lambda))^p
\]

from which it is seen that all the structure functions, and (56) also, will have maxima at the same position, \(r/\lambda = 1.11\). This is consistent with the usual assumption that the inertial range is the same for all the structure functions, but probably not exactly true. The sixth order structure functions presented by Anselmet et al\(^1\) peak at about \(r/\lambda = 1.11\) if account is taken of the different compensation powers in the presented data.

In Fig.4 the compensated structure function calculated from (61) is plotted versus \(r/\lambda\) for \(p=3\) and 12 for the single value \(R_\lambda = 1000\) in order to see the effect of increasing order on the shape. Besides increasing the distance from the asymptote, increasing the order increases the curvature at the maximum, an effect which may be seen in experiments.

### III. ANALYSIS OF THE K62 HYPOTHESIS

In this section it is not assumed that mean dissipation is finite as Reynolds number tends to infinity. The analysis of Sec.II can be carried quite far without this assumption. Let \(U_o\) and \(L_o\) be outer scales without making use of \(\epsilon\). It is important that the outer scales not depend on \(\nu\). Outer variables are then defined by

\[
r_o = \frac{r}{L_o}, \quad v_o = \frac{v}{U_o}, \quad p_o = \frac{P}{\rho U_o^2}, \quad \tau = \frac{t U_o}{L_o}.
\]

Then (8) and (10) become

\[
\frac{\partial v_o}{\partial \tau} + v_o \cdot \nabla v_o = -\nabla p_o + R_o^{-1} \nabla^2 v_o,
\]

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\[ \nabla_o \cdot \mathbf{v}_o = 0 \tag{64} \]

where \( R_o = U_o L_o / \nu \). An outer expansion may be taken in the form
\[ \mathbf{v} = U \mathbf{v}_o, \quad \mathbf{v}_o = \mathbf{v}_{o,1} + R_o^{-1} \mathbf{v}_{o,2} + \cdots. \tag{65} \]

Now change variables, defining inner variables by
\[ \mathbf{r}_i = f_\alpha(R_o) \mathbf{r}_o, \quad \mathbf{v}_i = f_\beta(R_o) \mathbf{v}_o, \quad P_i = f_\beta(R_o)^2 P_o. \tag{66} \]

Equation (62) becomes
\[ f_\alpha^{-1} f_\beta \frac{\partial \mathbf{v}_i}{\partial r} + \mathbf{v}_i \cdot \nabla_i \mathbf{v}_i = -\nabla_i P_i + R_o^{-1} f_\alpha f_\beta \nabla_i^2 \mathbf{v}_i. \tag{67} \]

In order to retain the viscous term in the limit it is required that
\[ f_\beta = R_o f_\alpha^{-1}. \tag{68} \]

Clearly another condition is required to complete this. However proceed to the matching step without this condition. Paralleling the steps leading to (35) one finds the matching condition
\[ \mathbf{v}_{o,1}(\mathbf{r}_o) = R_o^{-1} f_\alpha \mathbf{v}_{i,1}(f_\alpha \mathbf{r}_o). \tag{69} \]

A finite limit requires \( f_\alpha = R_o^{1/(1+q)} \) with
\[ \mathbf{v}_{o,1} = \mathbf{V} r_q, \quad \mathbf{v}_{i,1} = \mathbf{V} r_q. \tag{70} \]

Therefore the relationship (66) between inner and outer variables becomes
\[ \mathbf{r}_i = R_o^{1/(1+q)} \mathbf{r}_o, \quad \mathbf{v}_i = R_o^{q/(1+q)} \mathbf{v}_o \] (consistent with (25) if \( q = 1/3 \)) and the implied inner length and velocity scales are \( L_i = L_o R_o^{-1/(1+q)} \) and \( U_i = U_o R_o^{-q/(1+q)} \). Thus power laws result even allowing for more general scaling. The value of \( q \) is undetermined by this analysis, therefore both \( \mathbf{V} \) and \( q \) can depend on the realization of the turbulent flow and are to be regarded as random variables. The result from (70) is that
\[ \mathbf{v} = U_o \mathbf{V} (r/L_o)^q \tag{71} \]

for \( r \) in the inertial range \( L_i << r << L_o \).

The K62 theory depends on the hypothesis that the instantaneous dissipation averaged over a sphere centered at \( \mathbf{x} \) should replace the mean dissipation in the K41 theory. The average dissipation over a sphere of radius \( l \) is defined by
\[ \epsilon_l = \frac{3}{4 \pi l^3} \int_{r<l} 2\nu D_{ij}(\mathbf{x} + \mathbf{r}) D_{ij}(\mathbf{x} + \mathbf{r}) d\mathbf{r} \tag{72} \]

where \( D_{ij} \) is the rate of strain tensor, which in light of its dependence on the argument \( \mathbf{x} + \mathbf{r} \), may be written
\[ D_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial r_j} + \frac{\partial v_j}{\partial r_i} \right). \tag{73} \]
With some manipulation and the use of the divergence theorem (72) may be written

$$
\epsilon_i = \frac{3\nu}{4\pi l^3} \int_{r<l} \frac{\partial v_i}{\partial r_j} \frac{\partial v_i}{\partial r_j} d\mathbf{r} + \frac{3\nu}{4\pi l^3} \int_{r=l} n_j v_j \frac{\partial v_i}{\partial r_i} dS , \tag{74}
$$

the second integral being over the surface of a sphere of radius \( l \). Since the integrands depend only on \( \mathbf{v} \) and \( \mathbf{r} \) the energy equation derived from (8) may be used to substitute for the integrand in the first integral, with the result

$$
\epsilon_i = -\frac{3}{4\pi l^3} \int_{r=l} n_i v_i (P + \frac{1}{2} v_i v_i) dS + \frac{3\nu}{4\pi l^3} \int_{r=l} n_j v_i \left( \frac{\partial v_i}{\partial r_j} + \frac{\partial v_j}{\partial r_i} \right) dS - \frac{3}{4\pi l^3} \frac{\partial}{\partial t} \int_{r<l} \frac{1}{2} v_j v_j d\mathbf{r} . \tag{75}
$$

Now evaluate each of the integrals when the radius \( l \) is in the inertial range, where \( \mathbf{v} = U_o \mathbf{V}(l/L_o)^q \) may be used. The first integral is proportional to \((U_o^3/L_o)(l/L_o)^{3q-1}\), the second to \((U_o^3/L_o)R_o^{-1}(l/L_o)^{2q-1}\) and may be neglected compared to the first as \( R_o \to \infty \). The third integral, a volume integral, may be estimated using \(|\mathbf{v}| < U_o |\mathbf{V}| (r/L_o)^q \) for \( r < l \), with the result that it is smaller than \((U_o^3/L_o)(l/L_o)^{2q}\) and therefore, because \( l/L_o << 1 \), it is much smaller than the first integral when \( q < 1 \).

Therefore we have the result that

$$
\epsilon_r = C^3 \frac{U_o^3}{L_o^3} \left( \frac{r}{L_o} \right)^{3q-1} . \tag{76}
$$

where \( C \) is another random variable. If \( q = 1/3 \), \( \epsilon_r \) is independent of \( r \), and there is no inertial range intermittency. (An intuitive intermittency model consists of alternating stripes of zero and non-zero dissipation. When the averaging sphere is large compared to the scale of the stripes, the value of the average becomes insensitive to its size.) On the other hand if \( q < 1/3 \), \( \epsilon_r \) becomes larger for smaller \( r \) and dissipation becomes more concentrated at smaller scales, implying intermittency at inertial scales.

From (76) it may be seen that

$$
(r \epsilon_r)^{1/3} = C U_o(r/L_o)^q . \tag{77}
$$

Comparing this with (71) gives the result that

$$
\frac{\mathbf{v}}{(r \epsilon_r)^{1/3}} = \frac{\mathbf{V}}{C} \tag{78}
$$

is a random variable independent of \( r \) and \( R_o \), which is the main K62 hypothesis. However, the result (76) is very restrictive. The power law form limits the statistics one might prescribe for this quantity since, while \( C \) and \( q \) can be random variables they must be independent of \( r \). Kolmogorov, with no restrictions on the form of the function, assumed that the logarithm of \( \epsilon_r \) was normally distributed with variance linear in \( \ln(r) \), which is inconsistent with (76). Fractal statistics are also inconsistent with (76), since the pdf of
fractal sets depends on \( r \). In fact (76) is so restrictive that it implies that \( q = 1/3 \). To prove this first take the ensemble average of (72), taking the ensemble average inside the integral. The mild assumption (used by Kolmogorov) that the dissipation is homogeneous over regions of radius \( r \) gives \( < \epsilon_r > = \epsilon \). Then the ensemble average of (76) results in

\[
\epsilon = \frac{U^3}{L_o} \langle C^3 \left( \frac{r}{L_o} \right)^{3q-1} \rangle . \tag{79}
\]

Because \( \epsilon \) is independent of \( r \), the right-hand-side of (79) must also be independent of \( r \), and this can only be true if \( q = 1/3 \). This is an important conclusion which means that the mean dissipation is finite as \( R_\lambda \to \infty \) as assumed in sect.\( \Pi \); therefore the scaling \( U_o = U \) and \( L_o = L = U^3/\epsilon \) may be used. More importantly it means that there can be no inertial range intermittency as \( R_\lambda \to \infty \).

However, there is more information in (79) for large but finite \( R_\lambda \). Equation (79) is supposed to be true as \( R_\lambda \to \infty \) with \( r \) in the inertial range, that is with \( r/\lambda = O(1) \). Therefore, using \( L \) and \( U \), write

\[
\epsilon = \frac{U^3}{L} \langle C^3 \left( \frac{r}{\lambda L} \right)^{3q-1} \rangle , \tag{80}
\]

or, since \( \epsilon = U^3/L \) now, and \( \lambda/\lambda = 15/R_\lambda \),

\[
1 = \langle C^3 \exp((3q - 1)(\ln(r/\lambda) + \ln(\lambda/L))) \rangle . \tag{81}
\]

In order to have a finite limit as \( R_\lambda \to \infty \), with \( r/\lambda = O(1) \), it is necessary that \( (3q - 1) \ln(\lambda/L) = 3a \) as \( R_\lambda \to \infty \), with a function \( a \) which could be a constant, or could be a function of \( R_\lambda \) which tends to zero as \( R_\lambda \to \infty \). The result is

\[
q = \frac{1}{3} - \frac{a}{\ln(R_\lambda/15)} , \tag{82}
\]

which says that \( q \) could very slowly decay to \( 1/3 \) as \( R_\lambda \to \infty \), if \( a \) is only a constant. If the decay is as slow as this, the conclusions of this paper would not be in serious conflict with K62. For example the intermittency exponent \( \mu \) defined by \( < \epsilon_r^2 > \sim (r/L)^{-\mu} \) would be given by \( \mu = 2 - 6q = 6a/\ln(R_\lambda/15) \), using (76) and (82). If \( a = .25 \) this gives \( \mu = .36 \) for \( R_\lambda = 10^3 \) and \( \mu = .23 \) for \( R_\lambda = 10^4 \), values which are near the accepted value\(^{19} \)

\( \mu = .25 \pm .05 \).

**IV. CONCLUSIONS AND DISCUSSION**

The main result of this research is expressed by (55), restated below with \( \Delta v_r \equiv \mathbf{v}_c \cdot \hat{\mathbf{r}} \):

\[
\frac{\Delta v_r}{(er)^{1/3}} = V(1 - R^{-2/3}_\lambda \phi(r/\lambda)) \quad \text{as} \quad R_\lambda \to \infty \tag{83}
\]

where \( f(x) = Ax^{2/3} + Bx^{-4/3} \), \( V, A, B \) being random variables independent of \( r \) and \( R_\lambda \) (\( A \) and \( B \) assumed positive). The Taylor microscale \( \lambda \) is an intermediate variable with
properties \( \lambda/L = \sqrt{15 R^{1/2}_\lambda} \to 0 \) and \( \lambda/\eta = \sqrt{15 R^{1/4}_\lambda} \to \infty \) as \( R_\lambda \to \infty \); it is implied in (80) that \( r/\lambda = O(1) \). The statement (80) includes the often cited Kolmogorov second similarity hypothesis: When \( R_\lambda \) is sufficiently large \( \Delta v_r/(c_r)^{1/3} \) is a random variable independent of \( r \) and \( R_\lambda \) when \( r \) is in the inertial range \( \eta << r << L \). The compensating factor in (80) adds some precision to the statement.

The compensating factor has a maximum at a position proportional to \( \lambda \) which can be taken as a definition of the center of the inertial subrange. The compensating factor gives an analytical reason for the generally accepted\textsuperscript{13} experimental fact that \( \lambda \) lies in the inertial subrange. The speculative result expressed by (61) assumes that all the structure functions peak at the same value of \( \lambda \). It would be of value to see to what extent (61) describes the complete body of existing structure function data.

Equation (80) implies that the K41 structure functions are approached as \( R_\lambda \to \infty \), but slowly like \( R^{2/3}_\lambda \). This is in conflict with the K62 anomalous scaling results which give \( B_p \sim (r/L)^{\zeta_p} \) where the anomalous exponent \( \zeta_p \) differs significantly from the K41 value \( p/3 \). This was shown in the experiments of Anselmet et al\textsuperscript{12} and a large body of theoretical and experimental evidence, cited and reviewed by Frisch\textsuperscript{14}, Sreenivasan and Antonia\textsuperscript{15}, Nelkin\textsuperscript{16} and others, support it. It is thought that the discrepancy may be explained by the slow approach to the asymptote, which is likely to be even slower for higher order structure functions because of the requirement that \( pR^{2/3}_\lambda \) be small. There is experimental support for this conjecture. Antonia et al\textsuperscript{13} show that the anomalous difference \( p/3 - \zeta_p \) decreases with increasing \( R_\lambda \). In Fig.5, previously unpublished data of Marusic et al\textsuperscript{17}, show this trend for the sixth order anomalous difference. (This data is from a recent atmospheric boundary layer study in the Great Salt Desert at Dugway, Utah. The data have the property that the compensated structure functions have power law behavior for about a decade on the right of the maximum, the (log-log) slopes tending toward the horizontal as \( R_\lambda \) increases.) The very high \( R_\lambda \) atmospheric data\textsuperscript{18} shown in Fig.5 show a slower trend toward zero consistent with the slow decay indicated by (82), which gives \( \zeta_6 = 6q = 2 - 6a/\ln(R_\lambda/15) \), so the anomalous difference is \( 2 - \zeta_6 = 6a/\ln(R_\lambda/15) \), the same as \( \mu \). This is plotted in Fig.(5) with a \( = 0.25 \).

Frisch\textsuperscript{14} points out that the two basic empirical laws of fully developed turbulence, satisfied at least approximately for any turbulent flow, are the two-thirds law, and the law of finite energy dissipation in which the mean energy dissipation rate is independent of the viscosity in the limit as viscosity tends to zero. Both of these laws are “proved” in this paper, and can be shown to each imply the other. The connection is through Kolmogorov’s spherically averaged dissipation \( \epsilon_r \) and the \( r^q \) power law derived in this paper. Equation (76) shows that if \( q = 1/3 \) then \( \epsilon_r \) is constant, independent of both \( r \) and \( R_\lambda \). Therefore using local homogeneity \( \epsilon(= < \epsilon_r >) \) is independent of \( R_\lambda \). In the other direction (79) shows that since \( \epsilon \) is independent of \( r \), \( q \) must be 1/3. The implication of \( \epsilon_r \) being independent of \( r \), when \( r \) is in the inertial range, is that intermittency of dissipation is at scales below the inertial range. Therefore, according to the analysis of this paper, there can be no inertial range intermittency in the limit as Reynolds number tends to infinity, this being equivalent to finite dissipation in the limit. However, there could be inertial range intermittency which slowly fades out as \( R_\lambda \to \infty \), as suggested by (82).
ACKNOWLEDGMENTS

The author is indebted to a number of scholars for helpful discussions and suggestions; Dale Pullin, Mark Nelkin, Katepallli Sreenivasan, Erik Lindborg and Christos Vassilicos. I would also like to thank Yves Gagne for sending me third order structure function data, and my colleague Ivan Marusic for making previously unpublished sixth order structure function data available.


FIGURE CAPTIONS

Fig.1 Compensated third order structure function versus $r/\lambda$, from ref.3. $R_\lambda = 1000, 2000, 3000, 4000, 10000$ from bottom.

Fig.2 Maximum of compensated third order structure function versus $R_\lambda$. Data points Gagne$^{11}$, curve $0.8 - 8.45R_\lambda^{-2/3}$.

Fig.3 Compensated third order structure function versus $r/\lambda$ compared with Gagne$^{11}$ data for $R_\lambda = 2260$ (top curve) and $R_\lambda = 350$.

Fig.4 Compensated structure functions of 3rd order (top curve) and 12th order versus $r/\lambda$, for $R_\lambda = 1000$, using (61).

Fig.5 The anomalous difference, $2 - \zeta_6$, versus $R_\lambda$ for the 6th order structure function. Atmospheric data: Circles, Marusic et al$^{17}$; Triangle, Antonia et al$^{13}$; Square, Sreenivasan and Dhruva$^{18}$. Straight line, proportional to $R_\lambda^{-2/3}$, through the Marusic data suggests a trend toward a zero asymptote as $R_\lambda \to \infty$, however the extremely high $R_\lambda$ points$^{13,18}$ are fit better with the dashed line derived from (82).
Fig.1 Compensated third order structure function versus $r/\lambda$, from ref.3. $R_\lambda =$ 1000, 2000, 3000, 4000, 10000 from bottom.
Fig. 2 Maximum of compensated third order structure function versus $R_\lambda$. Data points Gagne$^{11}$, curve $0.8 - 8.45 R_\lambda^{-2/3}$.
Fig. 3 Compensated third order structure function versus $r/\lambda$ compared with Gagne\textsuperscript{11} data for $R_{\lambda} = 2260$ (top curve) and $R_{\lambda} = 350$. 
Fig. 4 Compensated structure functions of 3rd order (top curve) and 12th order versus $r/\lambda$, for $R_\lambda = 1000$, using (61).
Fig. 5 The anomalous difference, $2 - \zeta_6$, versus $R_\lambda$ for the 6th order structure function. Atmospheric data: Circles, Marusic et al\textsuperscript{17}; Triangle, Antonia et al\textsuperscript{13}; Square, Sreenivasan and Dhrula\textsuperscript{18}. Straight line, proportional to $R_\lambda^{-2/3}$, through the Marusic data suggests a trend toward a zero asymptote as $R_\lambda \to \infty$, however the extremely high $R_\lambda$ points\textsuperscript{13,18} are fit better with the dashed line derived from (82).