

Robust Control: Past Successes and Future Directions

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University of Minnesota



January 22, 2019

ONR / HONEYWELL WORKSHOP

Advances in
MULTIVARIABLE CONTROL

Lecture Notes
by
John Doyle

with contributions by
Cheng-Chih Chu
Bruce Francis
Prasad Khargonekar
Gunter Stein

MUSYN

ROBUST
MULTIVARIABLE
CONTROL:
THEORY
AND
APPLICATION
USING μ -TOOLS
AUGUST 4 - 7

MUSYN is pleased to announce the latest short course in robust multivariable control design. A detailed, five-day instructional workshop will be taught August 4-7 by three researchers in the field: Prof. John C. Doyle, Prof. Andy Packard and Prof. Gary J. Balas. The short course provides the attendees with an introduction to robust multivariable control using H_∞ and μ analysis and design techniques.

In the past three years over 200 people from industry, government laboratories and academia have attended this course. Locations have included: Los Angeles, Minneapolis, NASA Langley Research Center, Cambridge University, and Delft University, The Netherlands.

The course has been updated to reflect the latest advances in theory and software. The course covers: various models of uncertainty for components, motivation of "structured uncertainty models," analysis of effects of structured uncertainty using the structured singular value (μ), real/complex μ analysis, controller design using H_∞ and μ techniques, and example applications. Theoretical understanding of the subject material as well as its application to practical problems is emphasized.

Participants will learn and use the μ -Analysis and Synthesis Toolbox (μ -Tools) control design package in conjunction with MATLAB to apply the course material to application areas which include: flight control systems for advanced aircraft, space shuttle lateral axis control systems, and vibration attenuation of flexible structures. Each application lecture will discuss modeling of the physical system, formulation of the control problem, application of μ and H_∞ techniques and corresponding results. The participants will have an opportunity to analyze and design control laws for each example with the μ -Tools software following the talks.

μ -Analysis and Synthesis
TOOLBOX

For Use with MATLAB[®]

Gary J. Balas
John C. Doyle
Keith Glover
Andy Packard
Roy Smith

The MATH WORKS Inc.

Outline

- **Brief Overview of Robust Control**
- Robustness of Time-Varying Systems
- Future Directions
- Conclusions

Pillars of Robust Control

1. Multivariable Optimal Control

- H_2 , H_∞ , DK-synthesis

2. Fundamental Limitations of Control

- Bode sensitivity integral, complementary sensitivity integrals, constraints due to right-half plane poles and zeros.

3. Uncertainty Modeling and Robustness Analysis

- Linear Fractional Transformations (LFTs), Structured Singular Value (μ), Integral Quadratic Constraints (IQCs)

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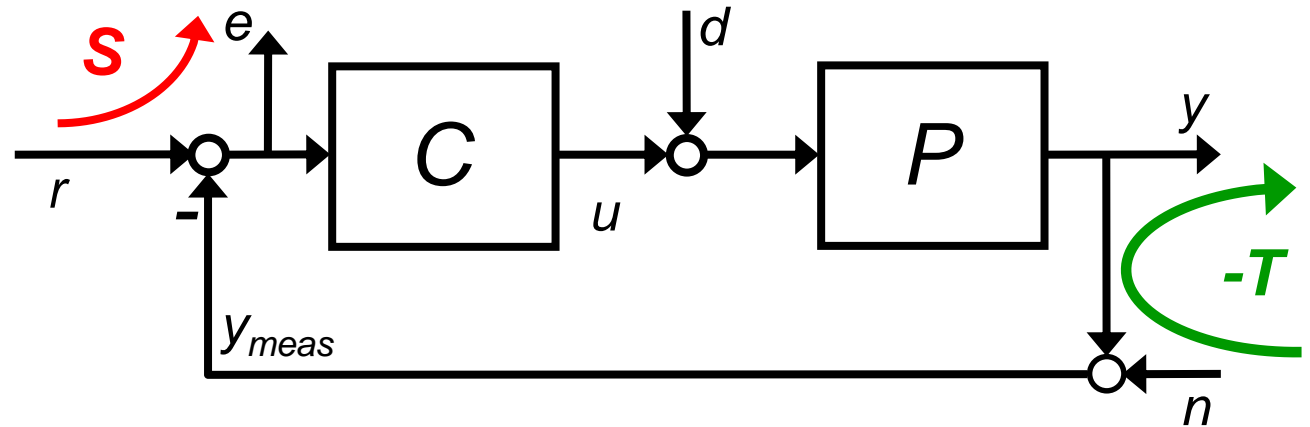
Basic Feedback Loop

Sensitivity

$$S = \frac{1}{1+PC}$$

Complementary Sensitivity

$$T = \frac{PC}{1+PC}$$



Many design objectives: Stability, disturbance rejection, reference tracking, noise rejection, moderate actuator commands, adequate robustness margins.

Basic Limitation: $S+T=1$

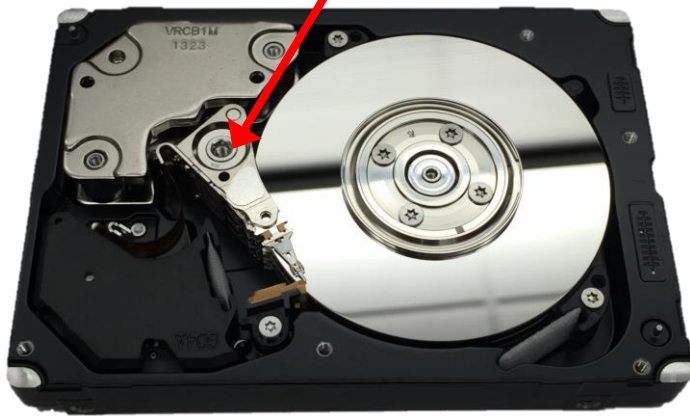
Typically require $|S| \ll 1$ at low frequencies for reference tracking and disturbance rejection.

Plant Uncertainty

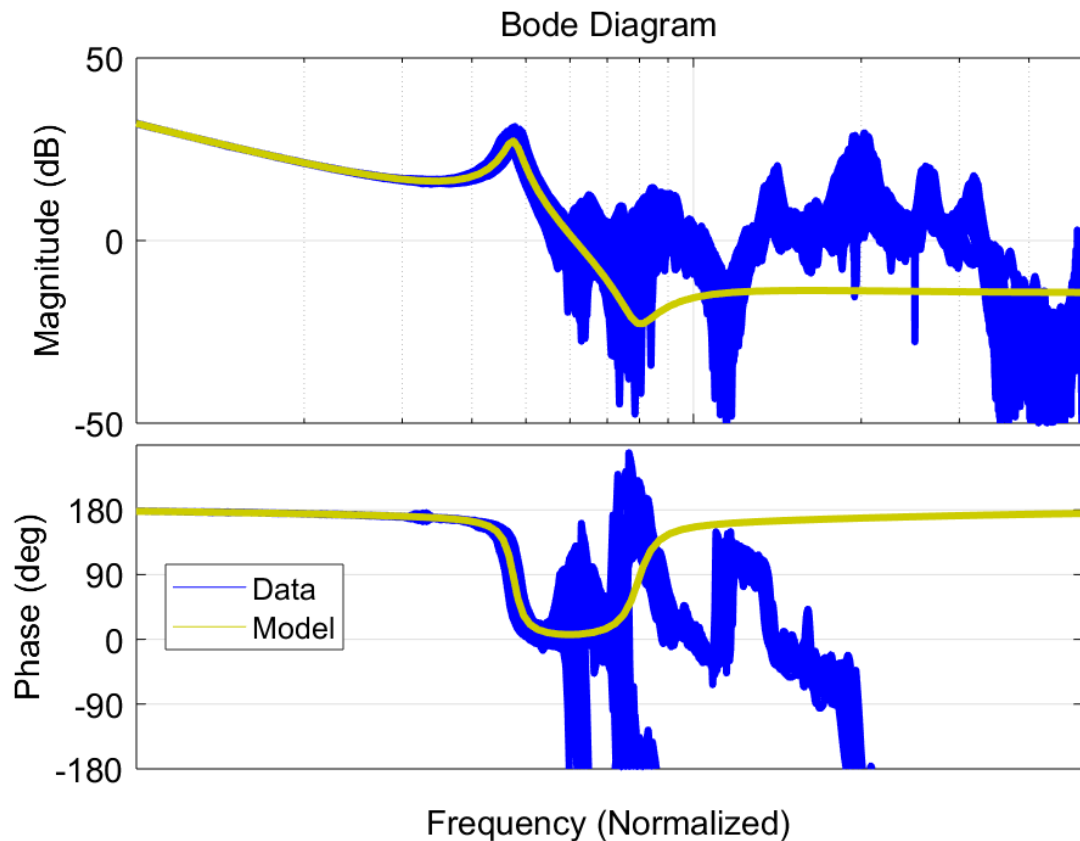
A simplified model P is used for control design.

- Actual dynamics are complex and have part-to-part variation.
- We lose model fidelity as we go to higher frequencies.

Voice Coil Motor



Experimental frequency responses (blue) and simplified model (black).



Stability Margins: Safety Factors for Control

Consider simple plant uncertainty types:

- Gain: αP where α varies from its nominal $\alpha_{nom}=1$
- Phase: $e^{j\theta}$ where θ varies from its nominal $\theta_{nom}=0$

Classical Margins: Largest gain/phase variations that can be tolerated before closed-loop instability occurs.

- Nyquist stability criterion is ideal for studying these margins
- Easily computed in SISO case, e.g. from frequency responses.

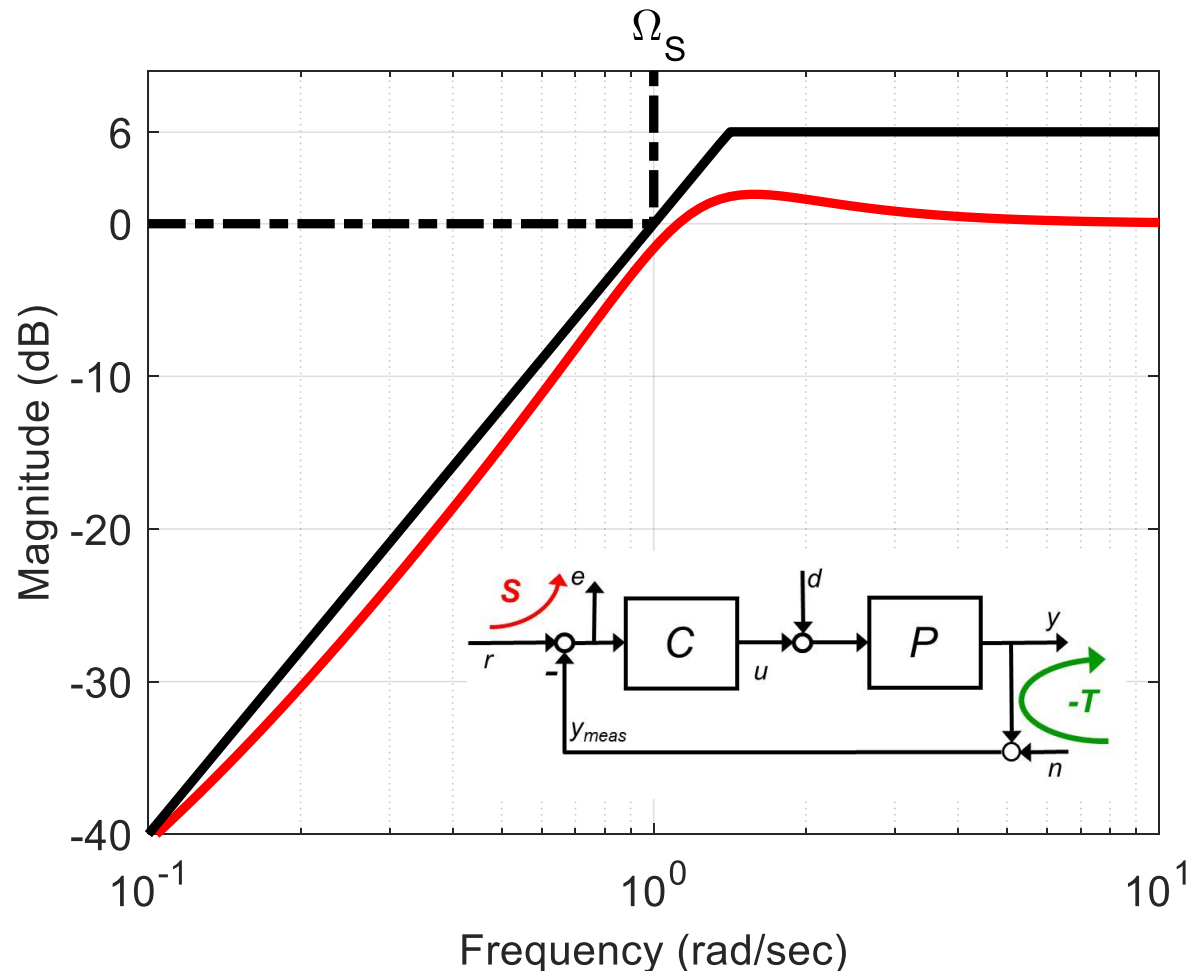
Disk Margin: If $|S(j\omega)|$ is large then a small (combined) gain/phase variation can cause instability.

$$|S(j\omega)| \gg 1 \Leftrightarrow |1 + P(j\omega)C(j\omega)| \approx 0$$

Typical Sensitivity Objectives

- **Performance:** “Small” $|S|$ up to 0 dB bandwidth Ω_S
- **Robustness:** $|S| \leq 2$ (=6dB) at all frequencies (No Peaks)

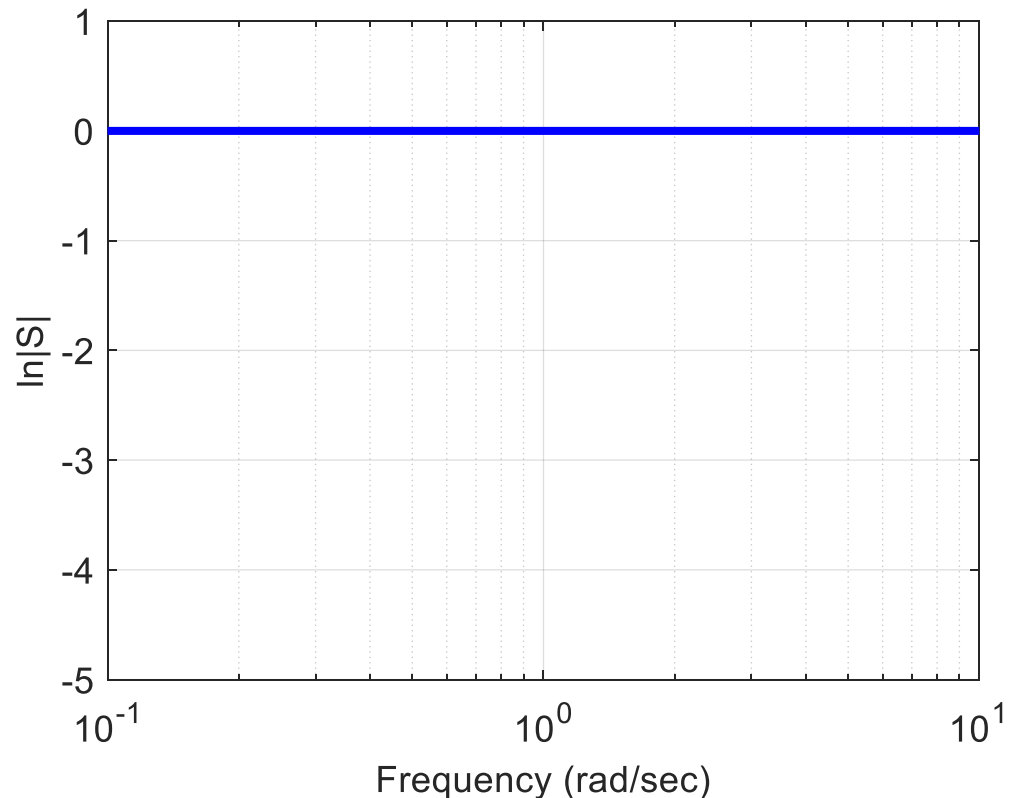
Typical sensitivity response (red) and design objectives (black)



Conservation of Sensitivity

Suppose P is stable so that $C = 0$ is a stabilizing controller.

$$S(s) := \frac{1}{1 + P(s)C(s)} \equiv 1 \quad \text{AND} \quad \int_0^{\infty} \ln |S(j\omega)| d\omega = 0$$



Conservation of Sensitivity

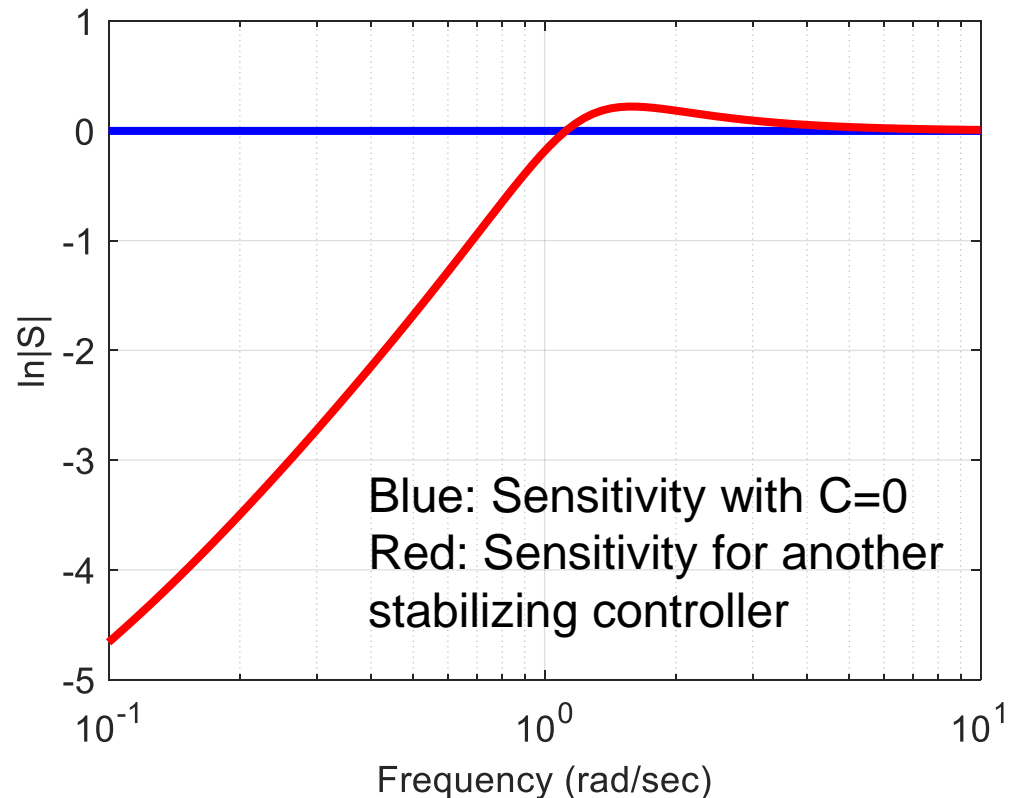
Suppose P is stable so that $C = 0$ is a stabilizing controller.

$$S(s) := \frac{1}{1 + P(s)C(s)} \equiv 1$$

AND $\int_0^{\infty} \ln |S(j\omega)| d\omega = 0$

Improving sensitivity at some frequencies leads to degradations at others.

This follows from the Bode Sensitivity Integral.



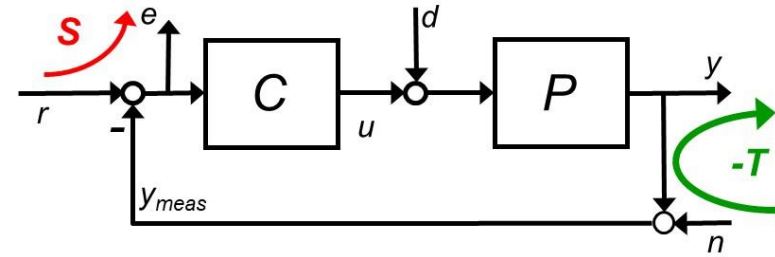
Bode Integral Theorem [1,2]

Assume PC has relative degree 2 and $S(s)$ is stable. Then:

$$\int_0^{\infty} \ln |S(j\omega)| d\omega = \pi \sum_{k=1}^{N_u} \operatorname{Re}(p_k) \geq 0$$

where p_k are the unstable (RHP) poles of PC .

(Note: $|S|$ (dB) $\approx 8.7 \ln |S|$)

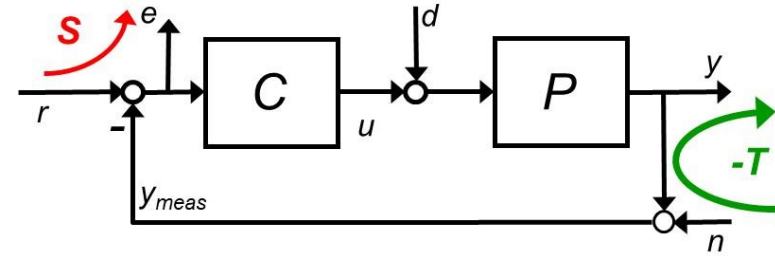


[1] Bode, Network Analysis and Feedback Amplifier Design, 1945.

[2] Freudenberg and Looze, Frequency Domain Properties of Scalar and Multivariable Feedback Systems, Springer-Verlag, 1988.

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(Note: $|S|$ (dB) $\approx 8.7 \ln |S|$)

This a key conserved quantity in feedback design.

Improving performance (e.g. increased bandwidth) comes at the expense of reduced robustness (peak in $|S|$) [3].

[1] Bode, Network Analysis and Feedback Amplifier Design, 1945.

[2] Freudenberg and Looze, Frequency Domain Properties of Scalar and Multivariable Feedback Systems, Springer-Verlag, 1988.

[3] Stein, Respect the Unstable, Bode Lecture, 1989 (and IEEE CSM, 2003)

Advanced Robustness Analysis

Move beyond classical SISO stability (gain/phase) margins

1. Multi-loop (MIMO) systems with multiple uncertainties
2. More detailed uncertainty descriptions including

<i>Structured Singular Value (μ)</i>	[• Parametric,]	<i>Integral Quadratic Constraints (IQCs)</i>
		• Non-parametric (dynamic)		
		• Nonlinearities, e.g. saturation		

3. Consider both robust stability and robust performance

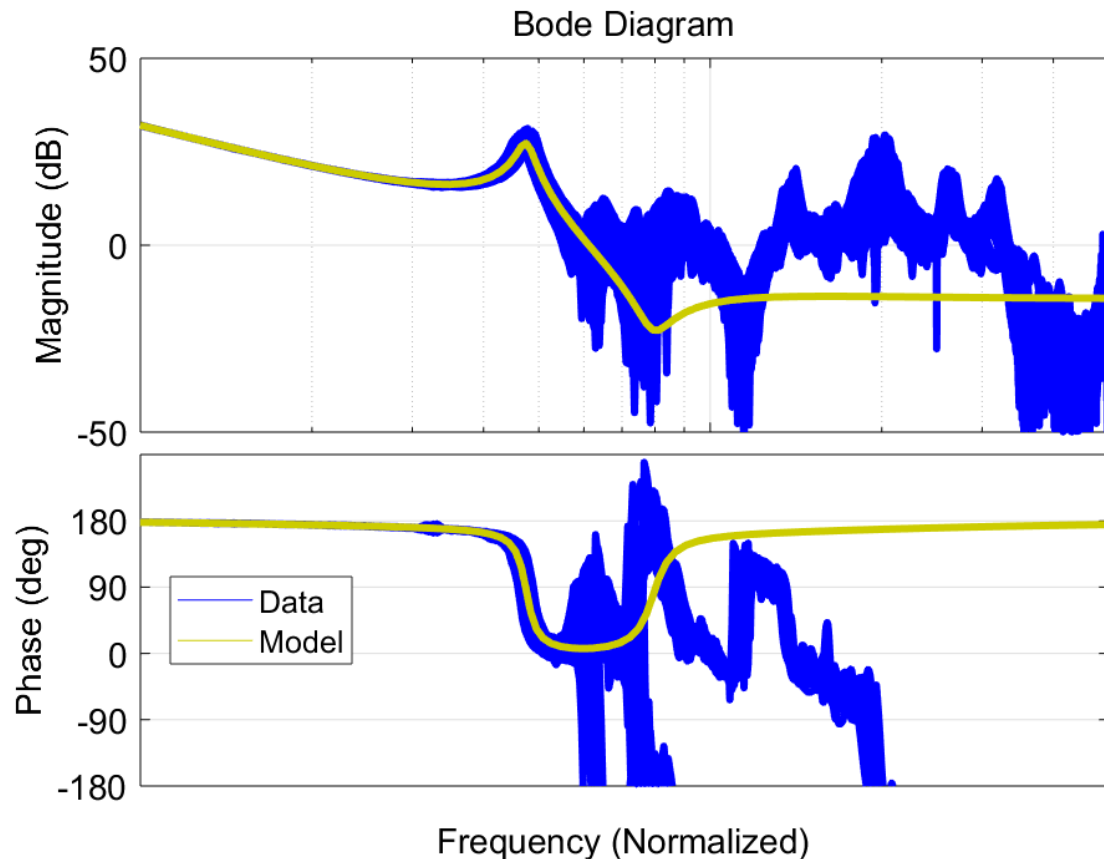
Developments go back to the Lur'e problem (40's) with key contributions in the 80's and 90's:

- μ : Safonov, Stein, Doyle, Packard, ...
- IQCs: Yakubovich, Megretski, Rantzer, ...

Non-parametric (Dynamic) Uncertainty

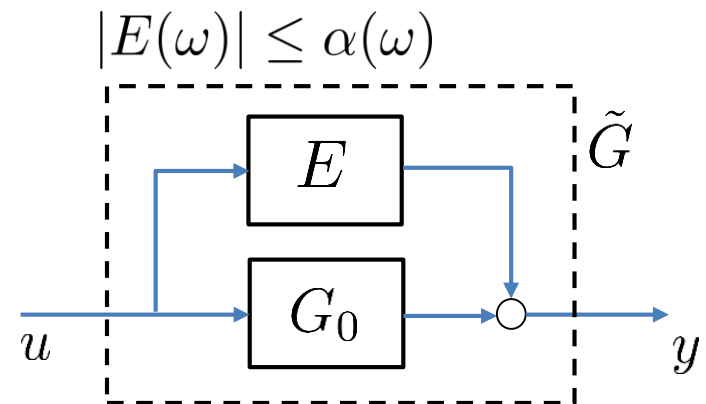
Model *nominal* behavior with LTI system G_0 .

Uncertainty modeled by LTI systems \tilde{G} close to G_0 in frequency response, e.g. small additive error.



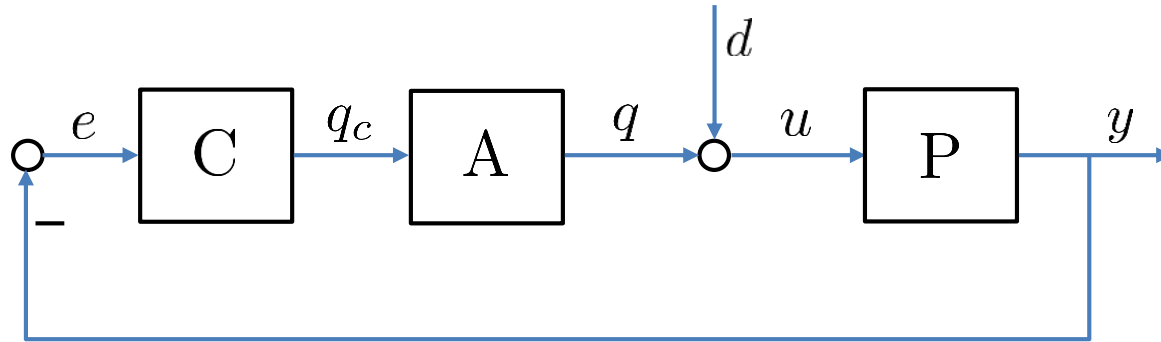
$$\underbrace{\left| \tilde{G}(\omega) - G_0(\omega) \right|}_{=: E} \leq \alpha(\omega)$$

Error Bound



Uncertainty Modeling

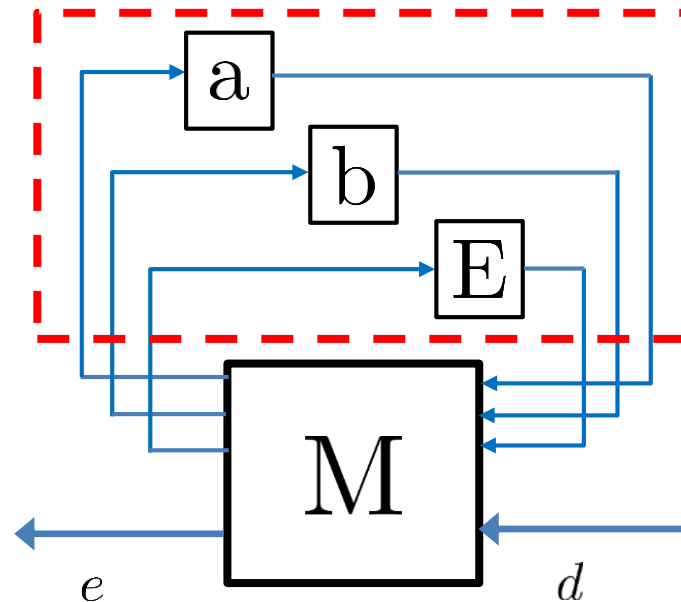
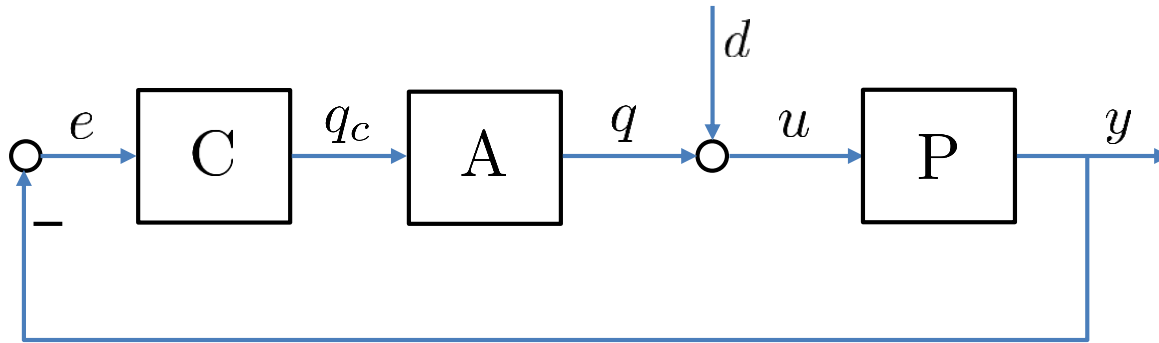
Consider SISO feedback system:



- Unstable plant with uncertain pole and input gain: $P(s) = \frac{b}{s+a}$ where $a \in [0.8, 1.1]$ and $b \in [1.7, 2.6]$
- First-order actuator with additive dynamic uncertainty $A(s) = A_0(s) + E(s)$ where $A_0(s) = \frac{10}{s+10}$ & $|E(\omega)| \leq 0.1$, E stable
- Proportional-Integral control $C(s) = \frac{3s+4.5}{s}$

Uncertainty Modeling

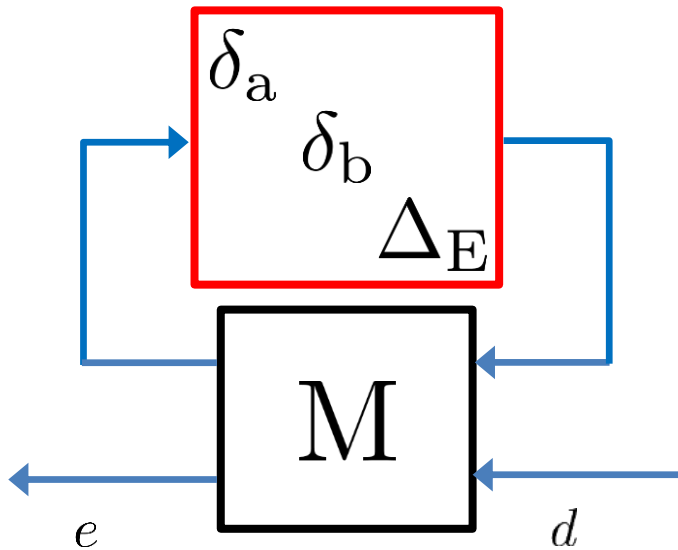
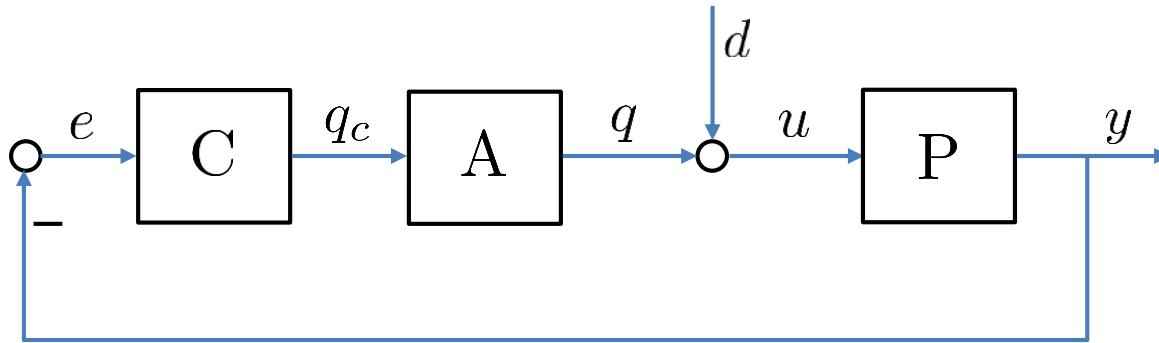
Separate known from the uncertain



Uncertainty is typically very structured

Uncertainty Modeling

Re-center and re-scale to normalize the uncertainties



Uncertainty set is structured:

$$\Delta := \{ \text{diag}(\delta_a, \delta_b, \Delta_E) : \delta_a, \delta_b \in \mathbb{R} \text{ and } \Delta_E \text{ LTI, stable} \}$$

where:

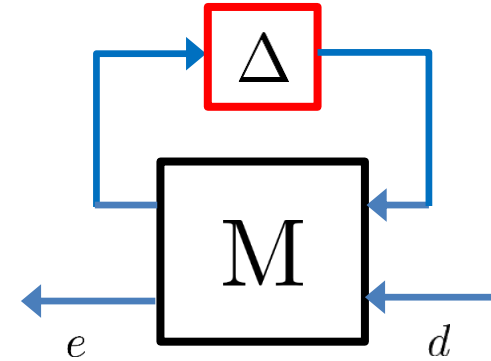
1. $\Delta=0$ gives nominal behavior
2. Range of modeled uncertainty is

$$\|\Delta\|_\infty := \sup_{\omega} \bar{\sigma}(\Delta) \leq 1$$

Robustness Metrics

Stability Margin: $\kappa_m := \inf_{\Delta \in \Delta} \|\Delta\|_\infty$
s.t. Δ causes instability

Worst-case Gain: $\sup_{\substack{\Delta \in \Delta, \\ \|\Delta\|_\infty \leq 1}} \|T_{d \rightarrow e}(M, \Delta)\|_\infty$



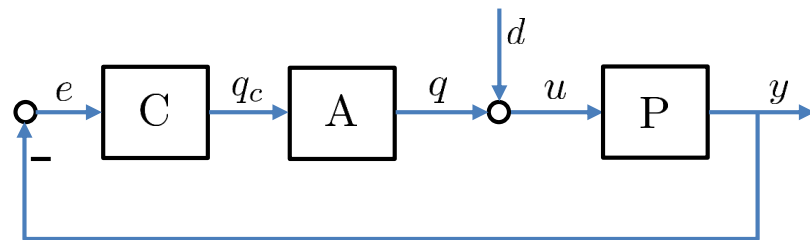
Comments:

- System is robustly stable if and only if $\kappa_m > 1$.
- Both metrics can be converted to a (freq. domain) μ test.
- Algorithms compute bounds that provide guarantees on performance and bad instances of uncertainties.
- IQCs extend the framework to include nonlinearities.

Numerical Algorithms and Software

Reliable software to create uncertainty models & perform analyses.

- Matlab's Robust Control Toolbox (Safonov & Chiang), (Balas, Doyle, Glover, Packard, & Smith), (Gahinet, Nemirovski, Laub, & Chilali)
- ONERA's Systems Modeling, Analysis and Control Toolbox (Biannic, Burlion, Demourant, Ferreres, Hardier, Loquen, & Roos)



Example Matlab code to assess robustness of simple feedback loop.

```
% Unstable plant with parametric uncertainty
a = ureal('a',1, 'Range', [0.8 1.1]);
b = ureal('b',2, 'Range', [1.7 2.6]);
P = tf(b, [1 -a]);

% Actuator with non-parametric (dynamic) unc.
nomAct = tf(10, [1 10]);
DeltaE = ultidyn('DeltaE',[1 1]);
A = nomAct + 0.1*DeltaE;

% Uncertain closed-loop (d->e) with PI control
C = tf([3 4.5],[1 0]);
R = feedback(-P, A*C);

% Robust stability and worst-case gain
[StabMargin, DestabilizingUncert] = robstab(R);
[wcGain, OffendingUncertainty] = wcgain(R);
```

Numerical Algorithms and Software

Reliable software to create uncertainty models & perform analyses.

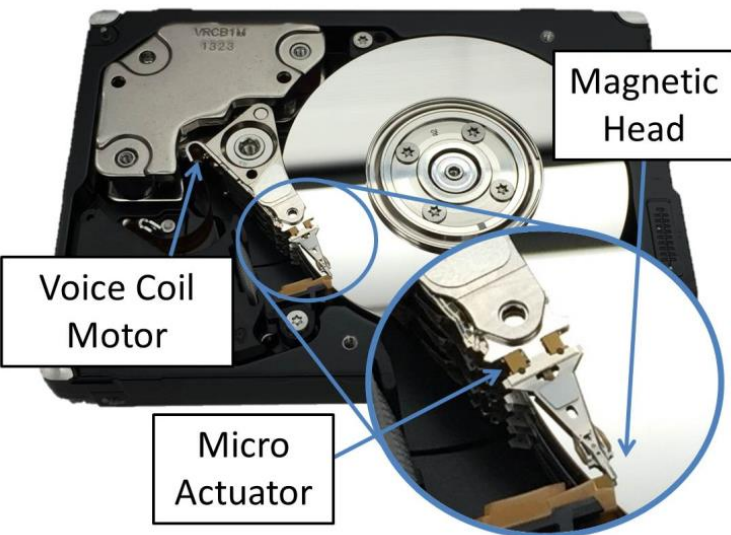
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Numerical algorithms continue to be developed, e.g. in Matlab:

- Structured H_∞ (R2010b) and `sys tune` (R2014a): Based on work by (Gahinet, Apkarian, Noll)
- μ without frequency gridding (R2016b): Based on work by (Gahinet, Balas, Packard, Seiler) and (Biannic, Ferreres, Roos)
- Automatic regularization for H_2 (R2017b) and H_∞ synthesis (R2018b): Based on work by (Gahinet, Packard, Seiler)

(My Recent) Applications of Robust Control

- Wind farm modeling and control (Annoni '16, Singh, Hoyt)
 - Individual turbine control (Wang '16, Ossmann, Theis)
 - UAV control with a single aerodynamic surface (Venkataraman '18)
 - Flexible aircraft (Kotikalpudi '17, Theis '18, Gupta, Pfifer)
 - Dual stage hard disk drives with Seagate (Honda '16)
- (Years refer to Ph.D. theses.)



Outline

- Brief Overview of Robust Control
- **Robustness of Time-Varying Systems**
 - **Joint work with M. Arcak, A. Packard, M. Moore, and C. Meissen at UC, Berkeley.**
 - **Funded by ONR BRC with B. Holm-Hansen at Tech. Monitor**
- Future Directions
- Conclusions

Time-Varying Systems



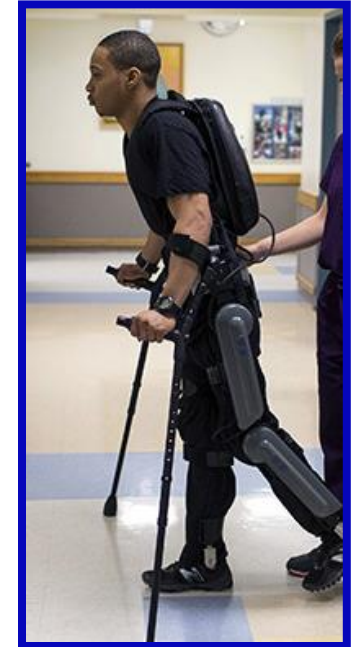
Wind Turbine
Periodic /
Parameter-Varying



Flexible Aircraft
Parameter-Varying



Vega Launcher
Time-Varying
(Source: ESA)



Robotics
Time-Varying
(Source: ReWalk)

Issue: Few numerically reliable methods to assess the robustness of time-varying systems.

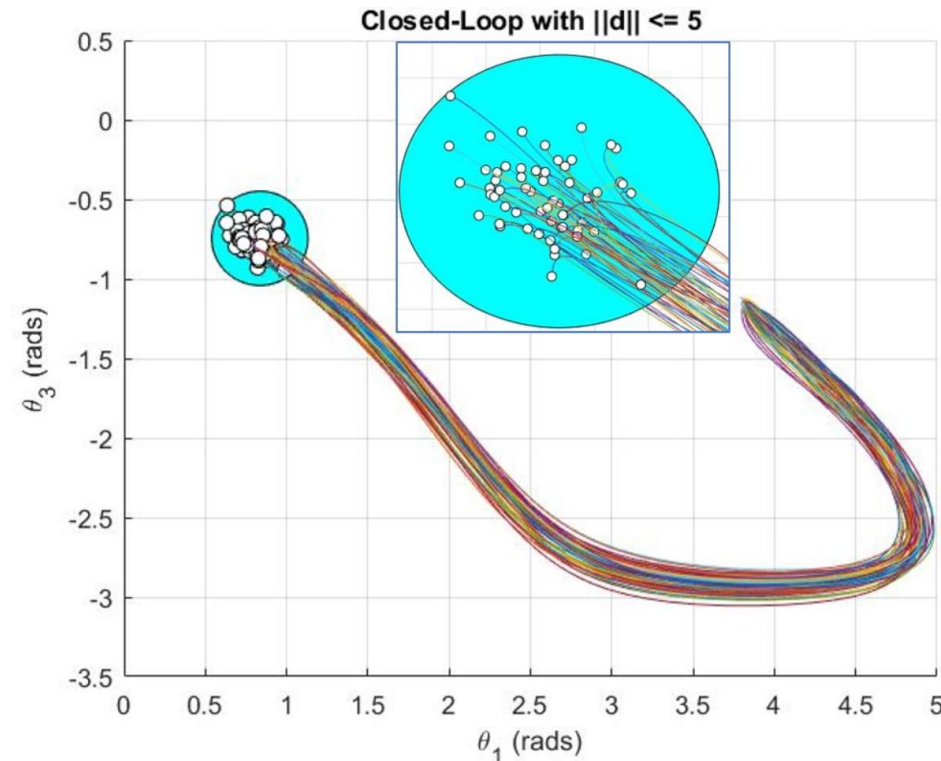
Finite Horizon Analysis

Goal: Assess the robustness of linear time-varying (LTV) systems on finite horizons.

Issue: Classical Gain/Phase Margins focus on (infinite horizon) stability and frequency domain concepts.

Instead focus on:

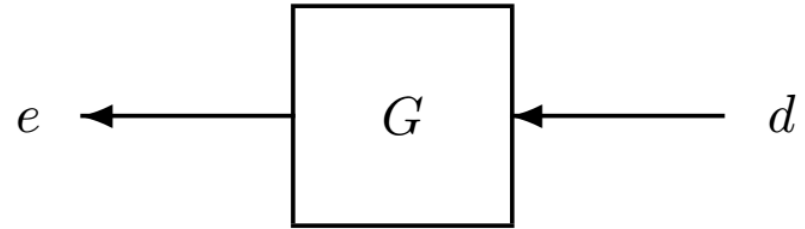
- Finite horizon metrics, e.g. induced gains and reachable sets.
- Effect of disturbances and (parametric, non-parametric and nonlinear) uncertainties.
- Time-domain analysis conditions.



(Nominal) Finite Horizon Analysis

Nominal LTV System

$$\begin{bmatrix} \dot{x}(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} A(t) & B(t) \\ C(t) & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \end{bmatrix}$$
$$x(0) = 0$$



Analysis Objective

Derive bound on $\|e(T)\|_2$ that holds for all disturbances $\|d\|_{2,[0,T]} \leq 1$ on the horizon $[0, T]$.

Nominal Analysis with Dissipation Inequalities

Theorem [1,2]

If there exists $P(\cdot) = P(\cdot)^T$ such that

(i) $P(T) = C(T)^T C(T)$, and

(ii) $V(x, t) := x^T P(t)x$ satisfies

$$\frac{d}{dt}V(x, t) - \gamma^2 d(t)^T d(t) \leq 0 \quad \forall t \in [0, T]$$

then $\|e(T)\|_2 \leq \gamma \|d\|_{2,[0,T]}$

Proof

Integrate dissipation inequality from $t = 0$ to $t = T$:

$$\underbrace{V(x(T), T)}_{=e(T)^T e(T)} - \underbrace{V(x(0), 0)}_{=0} - \gamma^2 \int_0^T d(t)^T d(t) dt \leq 0$$

[1] Moore, Finite Horizon Robustness Analysis using IQCs, MS Thesis, Berkeley, 2015.

[2] Willems, Dissipative Dynamical Systems: Parts i and ii, 1972.

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then $\|e(T)\|_2 \leq \gamma \|d\|_{2,[0,T]}$

Dissipation inequality can be recast as a differential LMI:

$$\begin{bmatrix} \dot{P} + A^T P + P A & P B \\ B^T P & -\gamma^2 I \end{bmatrix} \preceq 0 \quad \forall t \in [0, T]$$

[1] Moore, Finite Horizon Robustness Analysis using IQCs, MS Thesis, Berkeley, 2015.

[2] Willems, Dissipative Dynamical Systems: Parts i and ii, 1972.

Nominal Analysis with Dissipation Inequalities

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then $\|e(T)\|_2 \leq \gamma \|d\|_{2, [0, T]}$

Comments

- The dissipation inequality is equivalent to Riccati conditions [3] but enables extensions to robustness analysis.
- **Numerically reliable algorithm to construct worst-case disturbance [4].**

[1] Moore, Finite Horizon Robustness Analysis using IQCs, MS Thesis, Berkeley, 2015.

[2] Willems, Dissipative Dynamical Systems: Parts i and ii, 1972.

[3] Green & Limebeer, Linear Robust Control, 1995.

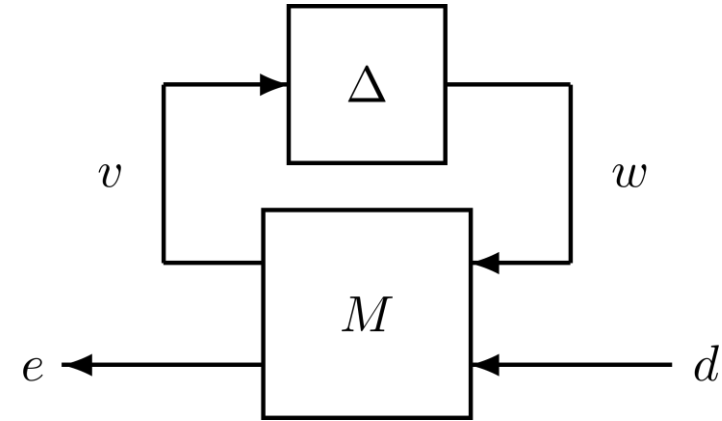
[4] Iannelli, Seiler, Marcos, “Construction of worst-case disturbances for LTV systems...”, 2019.

(Robust) Finite-Horizon Analysis

Uncertain LTV System

$$\begin{bmatrix} \dot{x}(t) \\ v(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} A(t) & B_1(t) & B_2(t) \\ C_1(t) & D_1(t) & D_2(t) \\ C_2(t) & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \\ d(t) \end{bmatrix}$$

$$x(0) = 0$$



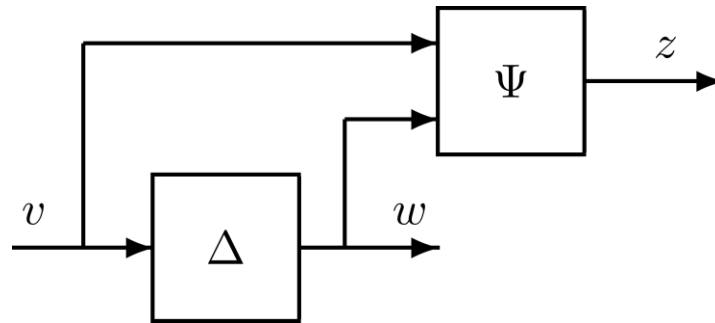
Uncertainty set Δ can be block-structured with parametric / non-parametric uncertainties and nonlinearities.

Analysis Objective

Derive bound on $\|e(T)\|_2$ that holds for all disturbances $\|d\|_{2,[0,T]} \leq 1$ and uncertainties $\Delta \in \Delta$ on the horizon $[0, T]$.

Integral Quadratic Constraints (IQCs) [1,2]

The robustness analysis uses constraints on the I/O behavior of Δ expressed as (time-domain) IQCs.



Definition

Δ satisfies the IQC on $[0, T]$ defined by a stable filter Ψ and matrix M if:

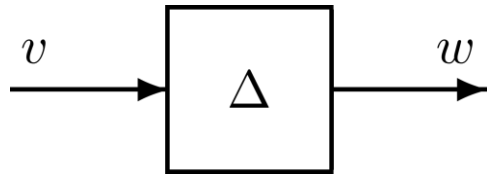
$$\int_0^T z(t)^T M z(t) dt \geq 0 \quad \forall v \in L_2[0, T] \text{ and } w = \Delta(v)$$

Notationally we consider only a single block. If Δ has block structure then we can specify different IQCs for each block.

[1] Yakubovich, S-procedure in nonlinear control theory, 1971.

[2] Megretski and Rantzer, System Analysis via Integral Quadratic Constraints, TAC, 1997.

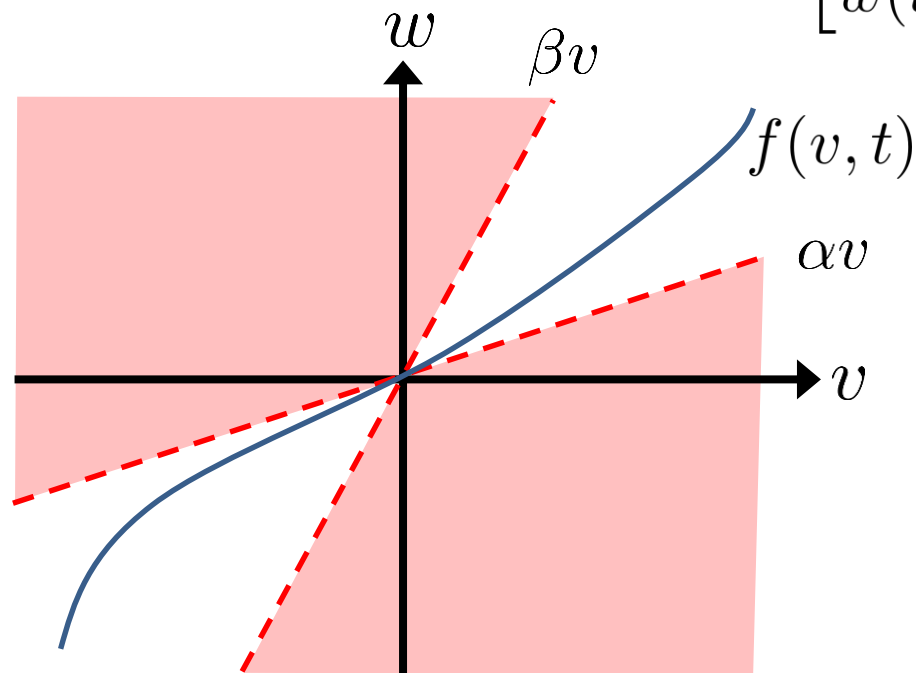
Example: Sector-bounded Nonlinearity



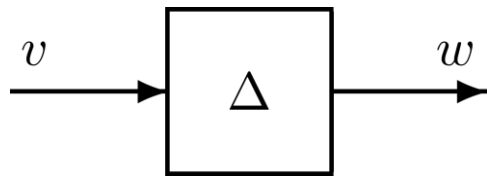
Δ is a sector-bounded nonlinearity, f .
 $(w(t) - \alpha v(t)) \cdot (\beta v(t) - w(t)) \geq 0 \quad \forall t$



$$\begin{bmatrix} v(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} -2\alpha\beta & \alpha + \beta \\ \alpha + \beta & -2 \end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} \geq 0 \quad \forall t$$



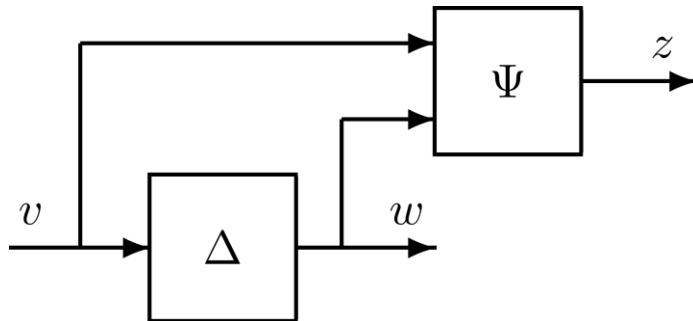
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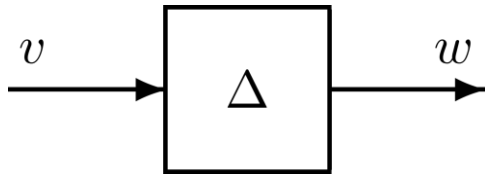
$$\begin{bmatrix} v(t) \\ w(t) \end{bmatrix}^T \underbrace{\begin{bmatrix} -2\alpha\beta & \alpha + \beta \\ \alpha + \beta & -2 \end{bmatrix}}_{:=M} \underbrace{\begin{bmatrix} v(t) \\ w(t) \end{bmatrix}}_{:=z(t)} \geq 0 \quad \forall t$$



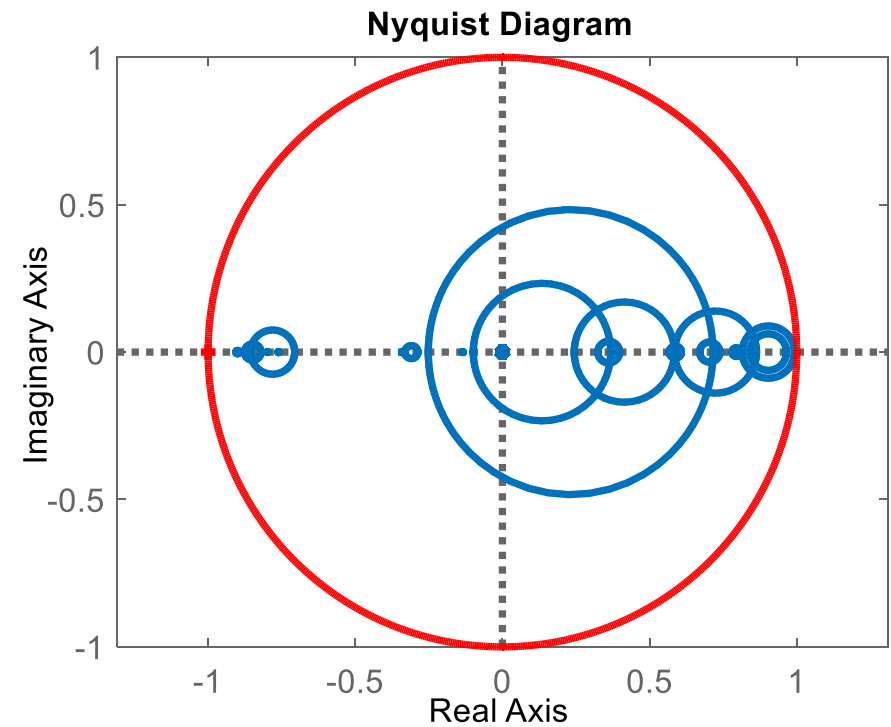
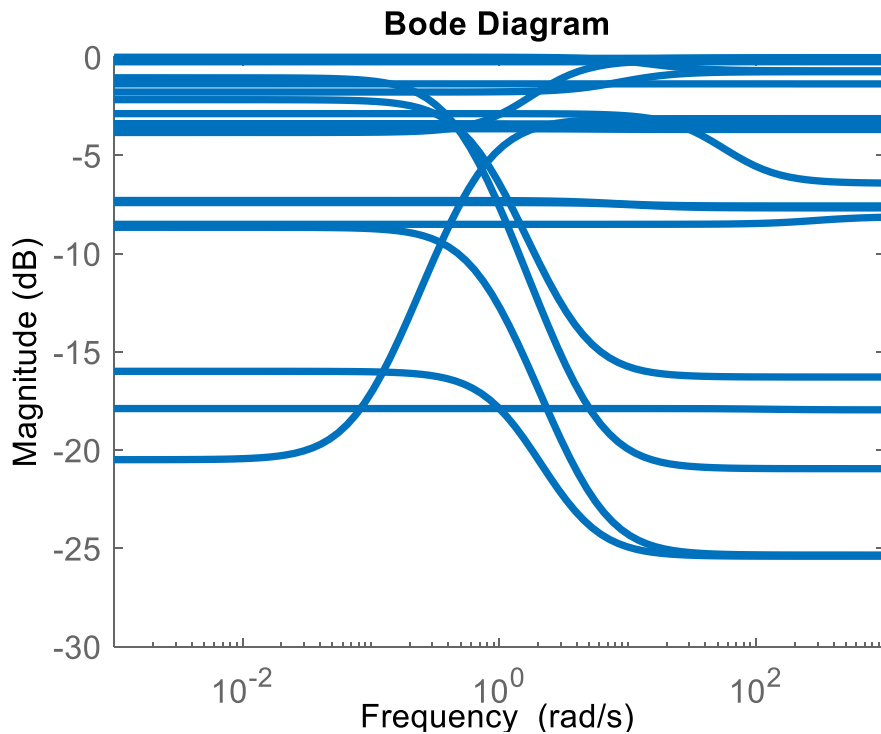
$$\int_0^T z(t)^T M z(t) dt \geq 0$$

Δ satisfies the IQC on $[0, T]$
 defined by $\Psi := I_2$ and M .

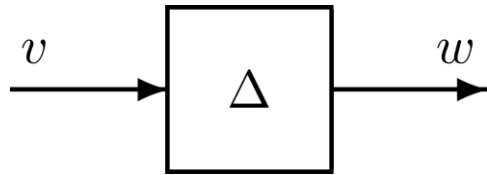
Example: Non-parametric (Dynamic) Uncertainty



Δ is stable, LTI with
 $\|\Delta\|_{\infty} := \sup_{\omega} |\Delta(\omega)| \leq 1$



Example: Non-parametric (Dynamic) Uncertainty



Δ is stable, LTI with
 $\|\Delta\|_\infty := \sup_\omega |\Delta(\omega)| \leq 1$

$$|\hat{w}(\omega)| \leq |\hat{v}(\omega)| \quad \forall \omega \quad \Rightarrow \quad \int_{-\infty}^{\infty} X(\omega) [|\hat{v}(\omega)|^2 - |\hat{w}(\omega)|^2] d\omega \geq 0$$

for any $X(\omega) \geq 0$

Frequency-Domain IQC \Leftrightarrow

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(\omega) \\ \hat{w}(\omega) \end{bmatrix}^* \begin{bmatrix} X(\omega) & 0 \\ 0 & -X(\omega) \end{bmatrix} \begin{bmatrix} \hat{v}(\omega) \\ \hat{w}(\omega) \end{bmatrix} d\omega \geq 0$$

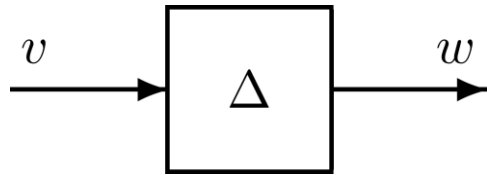
for any $X(\omega) \geq 0$

Spectral Factorization \Leftrightarrow

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(\omega) \\ \hat{w}(\omega) \end{bmatrix}^* \Psi(\omega)^* M \Psi(\omega) \begin{bmatrix} \hat{v}(\omega) \\ \hat{w}(\omega) \end{bmatrix} d\omega \geq 0$$

$X(\omega) = D(\omega)^* D(\omega)$ where
 $\Psi := \text{diag}(D, D)$ and $M := \text{diag}(1, -1)$

Example: Non-parametric (Dynamic) Uncertainty



Δ is stable, LTI with
 $\|\Delta\|_\infty := \sup_\omega |\Delta(\omega)| \leq 1$

$$|\hat{w}(\omega)| \leq |\hat{v}(\omega)| \quad \forall \omega \quad \Rightarrow \quad \int_{-\infty}^{\infty} X(\omega) [|\hat{v}(\omega)|^2 - |\hat{w}(\omega)|^2] d\omega \geq 0$$

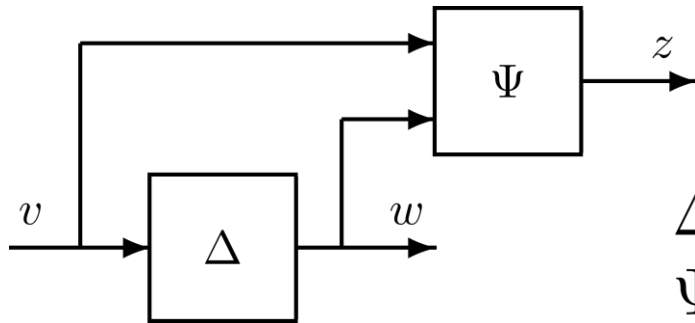
for any $X(\omega) \geq 0$

Frequency-Domain IQC \Leftrightarrow

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(\omega) \\ \hat{w}(\omega) \end{bmatrix}^* \begin{bmatrix} X(\omega) & 0 \\ 0 & -X(\omega) \end{bmatrix} \begin{bmatrix} \hat{v}(\omega) \\ \hat{w}(\omega) \end{bmatrix} d\omega \geq 0$$

for any $X(\omega) \geq 0$

(Invoke Causality)



$$\int_0^T z(t)^T M z(t) dt \geq 0$$

Δ satisfies the IQC on $[0, T]$ defined by
 $\Psi := \text{diag}(D, D)$ and $M := \text{diag}(1, -1)$

[1] Balakrishnan, Lyapunov Functionals in Complex μ Analysis, TAC, 2002.

Additional IQC Details

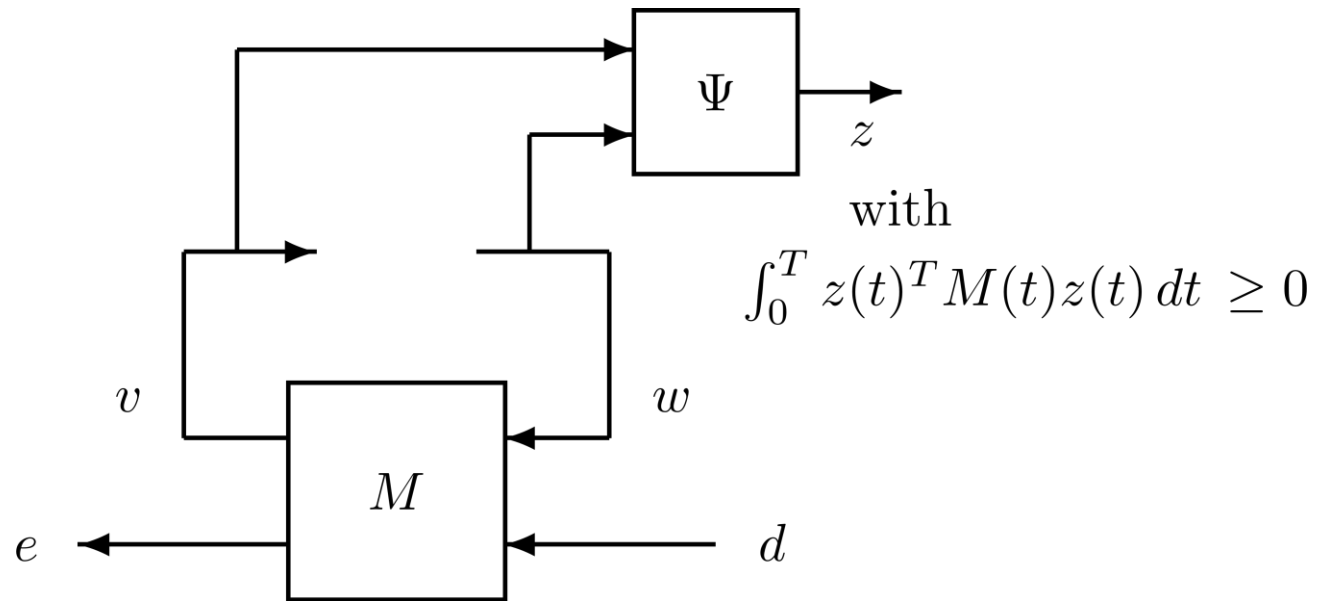
- Most IQCs are related to previous robust stability results
 - IQC for sector nonlinearities related to the circle criterion
 - IQC for LTI uncertainties related to D-scales in μ analysis
- A dictionary of additional IQC for various uncertainties / nonlinearities is given in [1].
 - IQCs for passive operators, static memoryless nonlinearities (Popov, Zames-Falb), time-delays, real parameters, etc.
 - Many IQCs are specified in the frequency domain
- **A technical J-spectral factorization result can be used to convert freq. domain IQCs into time-domain IQCs [2,3].**

[1] Megretski & Rantzer, System analysis via IQCs, TAC, 1997. [IQC derived based on much prior literature]

[2] Seiler, **Stability Analysis with Dissipation Inequalities and Integral Quadratic Constraints**, TAC, 2015.

[3] Hu, Lacerda, Seiler, **Robustness Analysis of Uncertain Discrete-Time System with ... IQCs**, IJRN, 2016.

Robustness Analysis



The robustness analysis is performed on the extended (LTV) system of (M, Ψ) using the constraint on z .

$$\begin{bmatrix} \dot{x}_e(t) \\ z(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A}(t) & \mathcal{B}_1(t) & \mathcal{B}_2(t) \\ \mathcal{C}_1(t) & \mathcal{D}_1(t) & \mathcal{D}_2(t) \\ \mathcal{C}_2(t) & 0 & 0 \end{bmatrix} \begin{bmatrix} x_e(t) \\ w(t) \\ d(t) \end{bmatrix}$$

Robust Finite Horizon Analysis

Theorem [1,2]

Assume Δ satisfies the IQC defined by (Ψ, M) .

If there exists $P(\cdot) = P(\cdot)^T$ such that

(i) $P(T) = \mathcal{C}_2(T)^T \mathcal{C}_2(T)$, and

(ii) $V(x, t) := x^T P(t)x$ satisfies

$$\frac{d}{dt}V(x, t) - \gamma^2 d(t)^T d(t) + z(t)^T M z(t) \leq 0 \quad \forall t \in [0, T]$$

then $\|e(T)\|_2 \leq \gamma \|d\|_{2, [0, T]}$

Proof

Integrate dissipation inequality from $t = 0$ to $t = T$:

$$\underbrace{V(x(T), T)}_{=e(T)^T e(T)} - \underbrace{V(x(0), 0)}_{=0} - \gamma^2 \int_0^T d(t)^T d(t) dt + \underbrace{\int_0^T z(t)^T M z(t) dt}_{\geq 0} \leq 0$$

[1] Moore, Finite Horizon Robustness Analysis using IQCs, MS Thesis, Berkeley, 2015.

[2] Seiler, Moore, Meissen, Arcak, Packard, Finite Horizon Robustness Analysis of LTV Systems Using IQCs, arXiv 2018 and Automatica 2019.

Robust Finite Horizon Analysis

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then $\|e(T)\|_2 \leq \gamma \|d\|_{2, [0, T]}$

Dissipation inequality can be recast as a differential LMI:

$$\begin{bmatrix} \dot{P} + \mathcal{A}^T P + P \mathcal{A} & P \mathcal{B}_1 & \mathcal{B}_2 \\ \mathcal{B}_1^T P & 0 & 0 \\ \mathcal{B}_2^T P & 0 & -\gamma^2 I \end{bmatrix} + (\cdot)^T M [\mathcal{C}_1 \quad \mathcal{D}_1 \quad \mathcal{D}_2] \preceq 0$$

$$\forall t \in [0, T]$$

[1] Moore, Finite Horizon Robustness Analysis using IQCs, MS Thesis, Berkeley, 2015.

[2] Seiler, Moore, Meissen, Arcak, Packard, Finite Horizon Robustness Analysis of LTV Systems Using IQCs, arXiv 2018 and Automatica 2019.

Numerical Algorithms and Software

- **Robustness Algorithms**

- Differential LMI can be “solved” via convex optimization using basis functions for $P(\cdot)$ and gridding on time [1].
- A more efficient algorithm mixes the differential LMI and a related Riccati Differential Equation condition [2].
- Similar methods developed for LPV [4,5] and periodic systems [6].

- **LTVTools Software [3]**

- Time-varying state space system objects, e.g. obtained from Simulink snapshot linearizations.
- Includes functions for nominal and robustness analyses.

[1] Moore, Finite Horizon Robustness Analysis using IQCs, MS Thesis, Berkeley, 2015.

[2] Seiler, Moore, Meissen, Arcak, Packard, Finite Horizon Robustness Analysis of LTV Systems Using IQCs, arXiv 2018 and Automatica 2019.

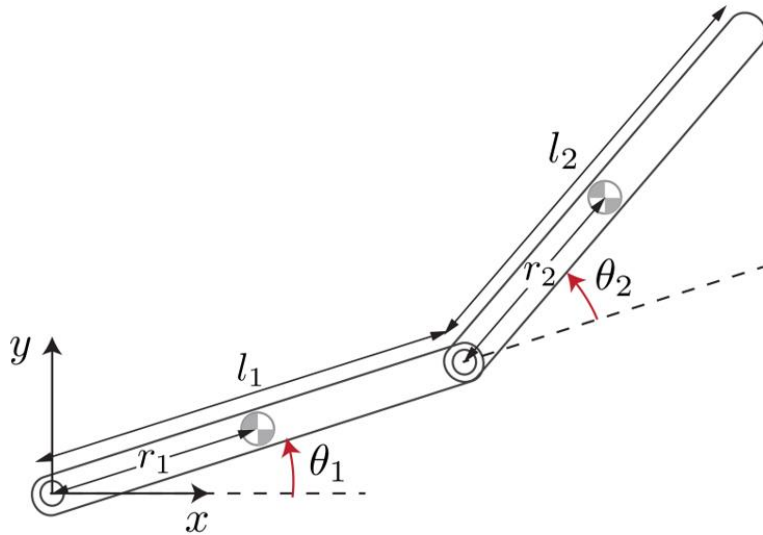
[3] <https://z.umn.edu/LTVTools>

[4] Pfifer & Seiler, Less Conservative Robustness Analysis of LPV Systems Using IQCs, IJRNC, 2016.

[5] Hjartarson, Packard, Seiler, LPVTools: A Toolbox for Modeling, Analysis, & Synthesis of LPV Systems, 2015.

[6] Fry, Farhood, Seiler, IQC-based robustness analysis of discrete-time LTV systems, IJRNC 2017.

Two-Link Robot Arm



Two-Link Diagram [1]

Nonlinear dynamics [MZS]:

$$\dot{\eta} = f(\eta, \tau, d)$$

where

$$\eta = [\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2]^T$$

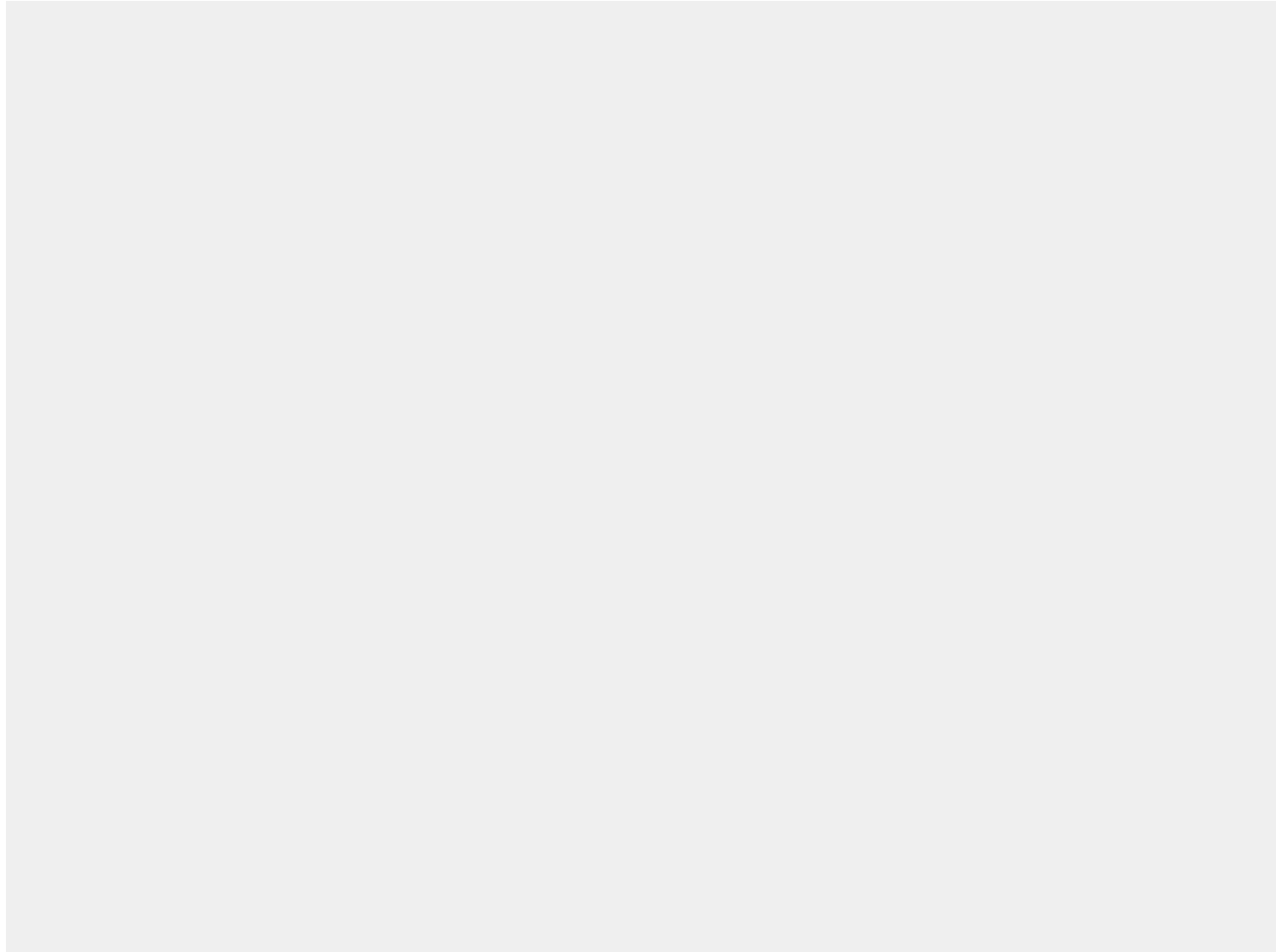
$$\tau = [\tau_1, \tau_2]^T$$

$$d = [d_1, d_2]^T$$

τ and d are control torques and disturbances at the link joints.

[1] R. Murray, Z. Li, and S. Sastry. *A Mathematical Introduction to Robot Manipulation*, 1994.

Nominal Trajectory in Cartesian Coordinates



Analysis

Nonlinear dynamics:

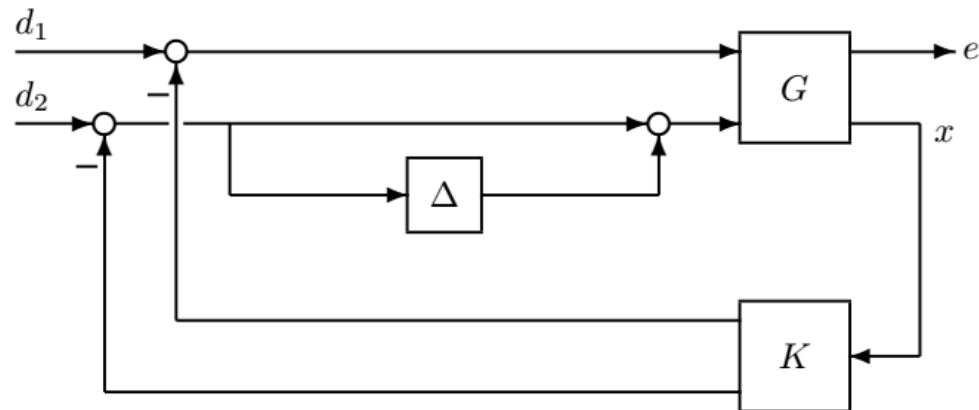
$$\dot{\eta} = f(\eta, \tau, d)$$

Linearize along the finite –horizon trajectory ($\bar{\eta}, \bar{\tau}, d = 0$)

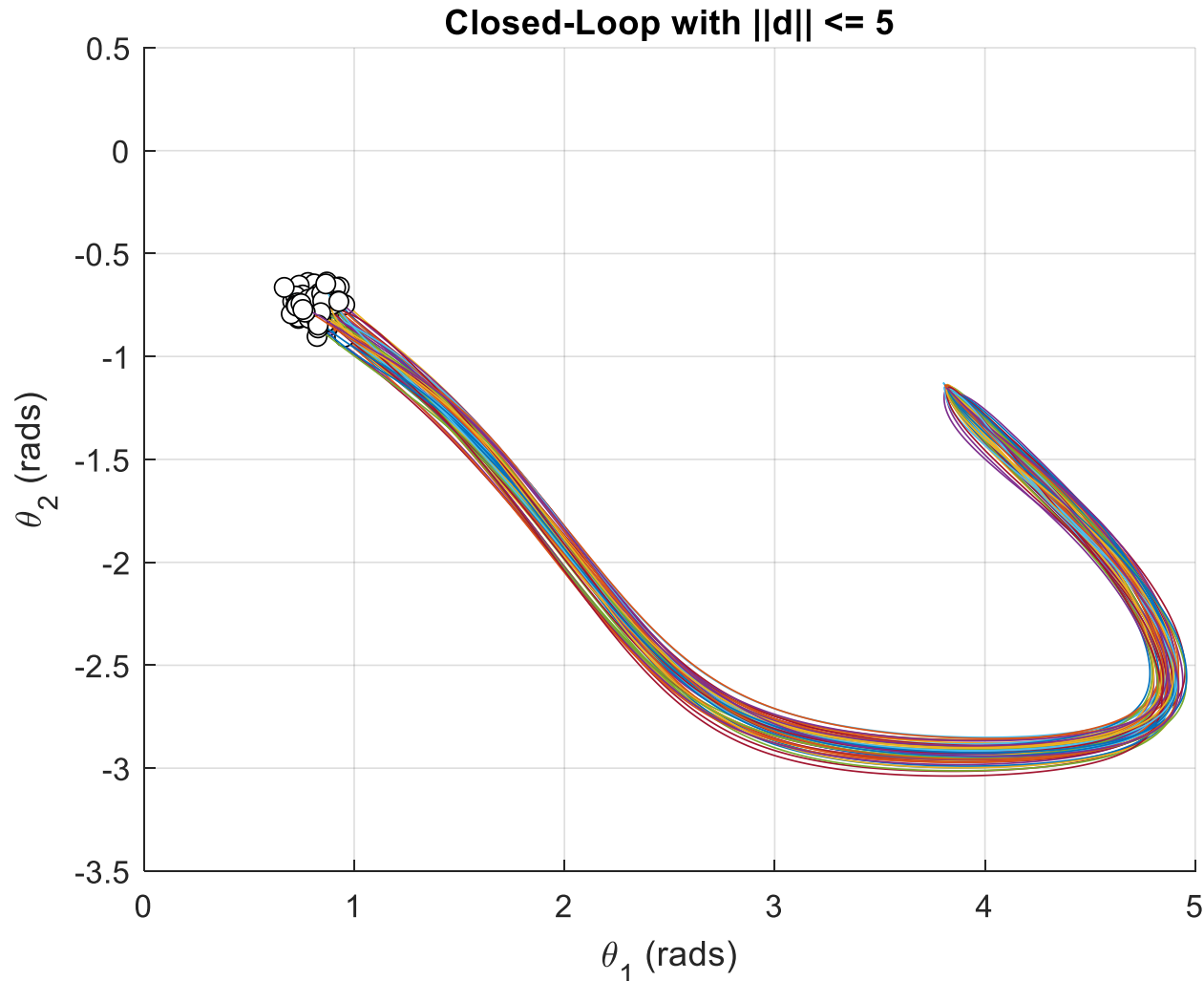
$$\dot{x} = A(t)x + B(t)u + B(t)d$$

Design finite-horizon state-feedback LQR gain.

Goal: Compute bound on the final position accounting for disturbances and LTI uncertainty Δ at 2nd joint.

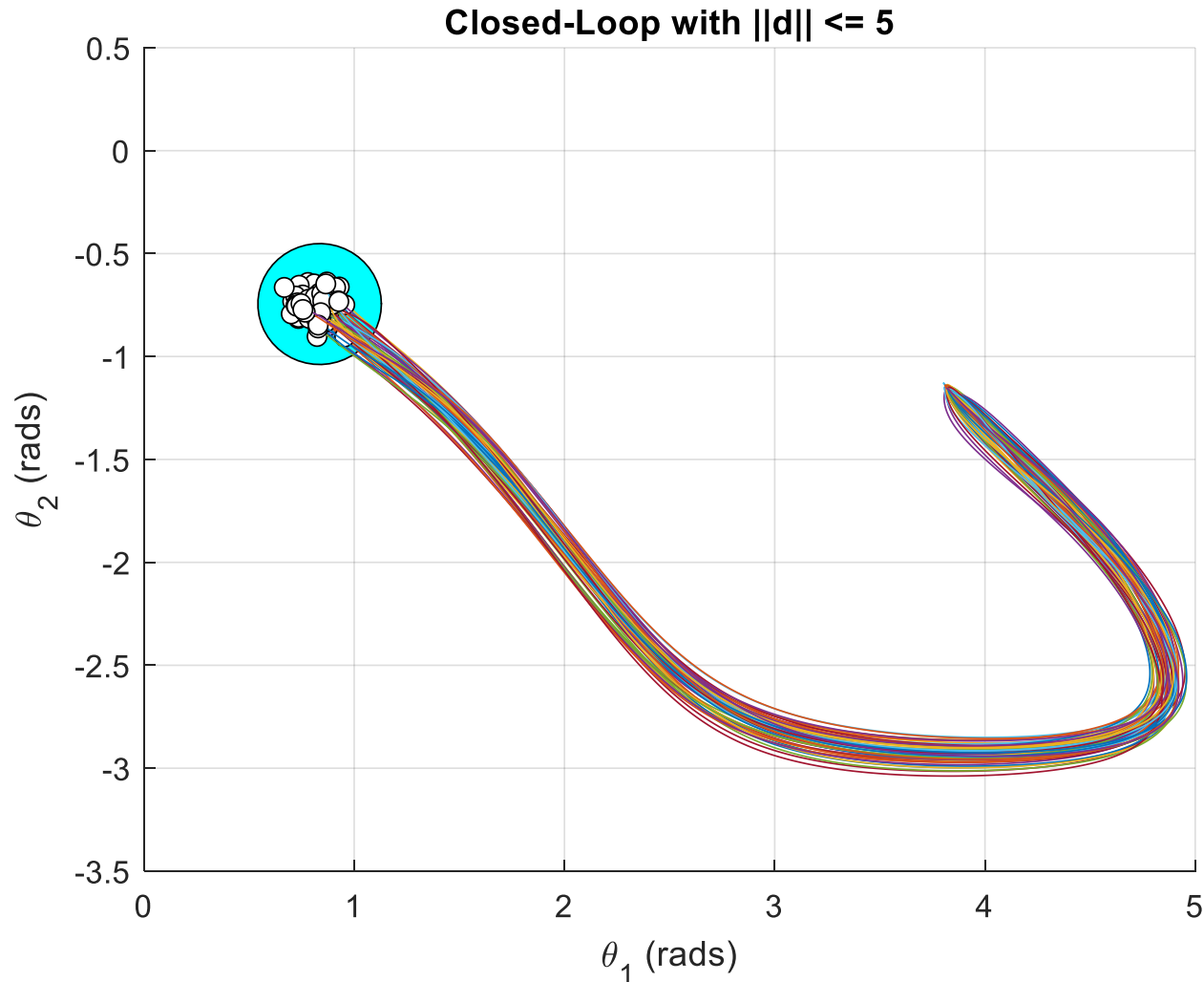


Monte-Carlo Simulations



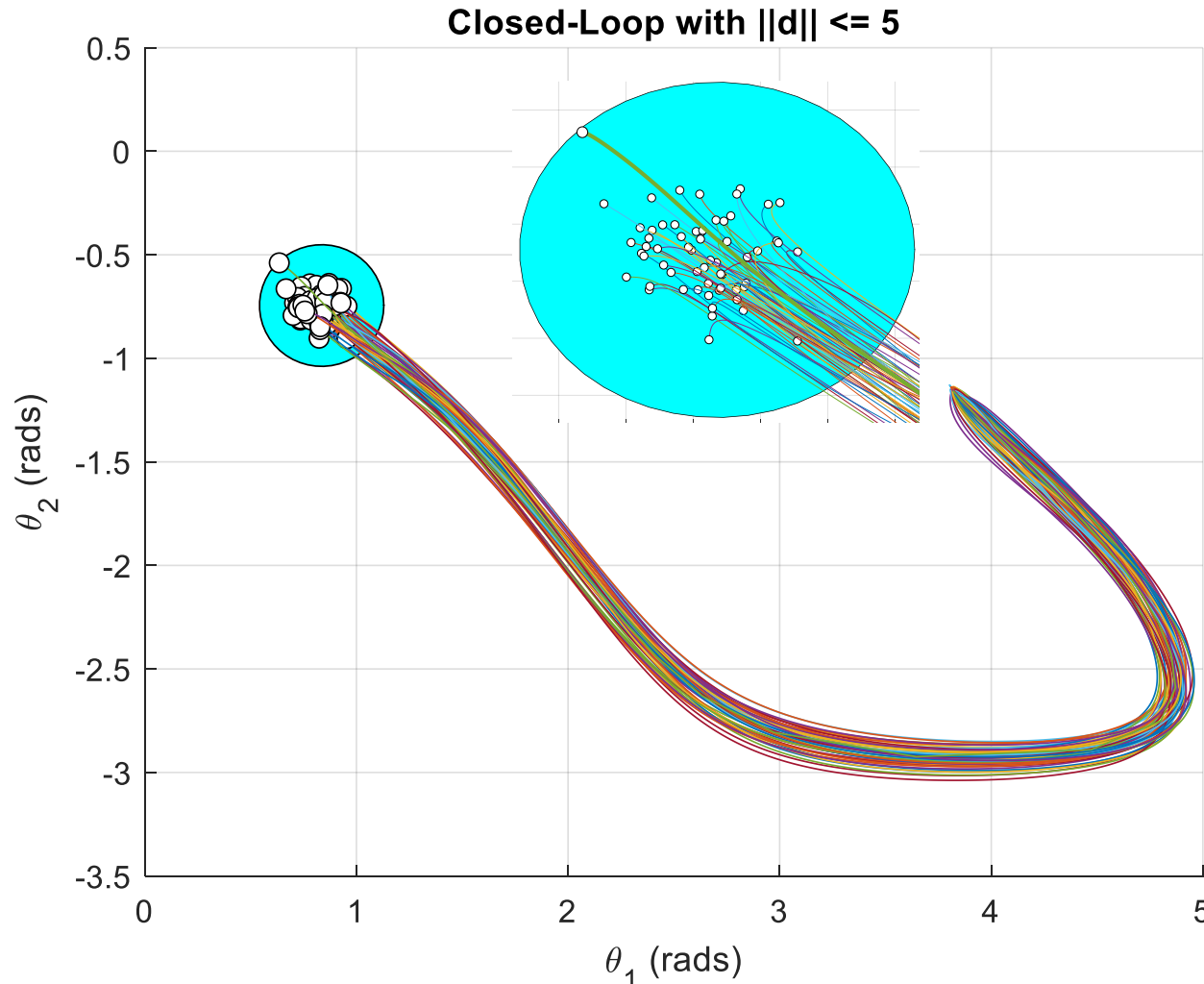
LTV simulations with randomly sampled disturbances and uncertainties (overlaid on nominal trajectory).

Robustness Bound



Cyan disk is bound computed in 102 sec using IQC/DI method
Bound accounts for disturbances $\|d\| \leq 5$ and $\|\Delta\| \leq 0.8$

Worst-Case Uncertainty / Disturbance



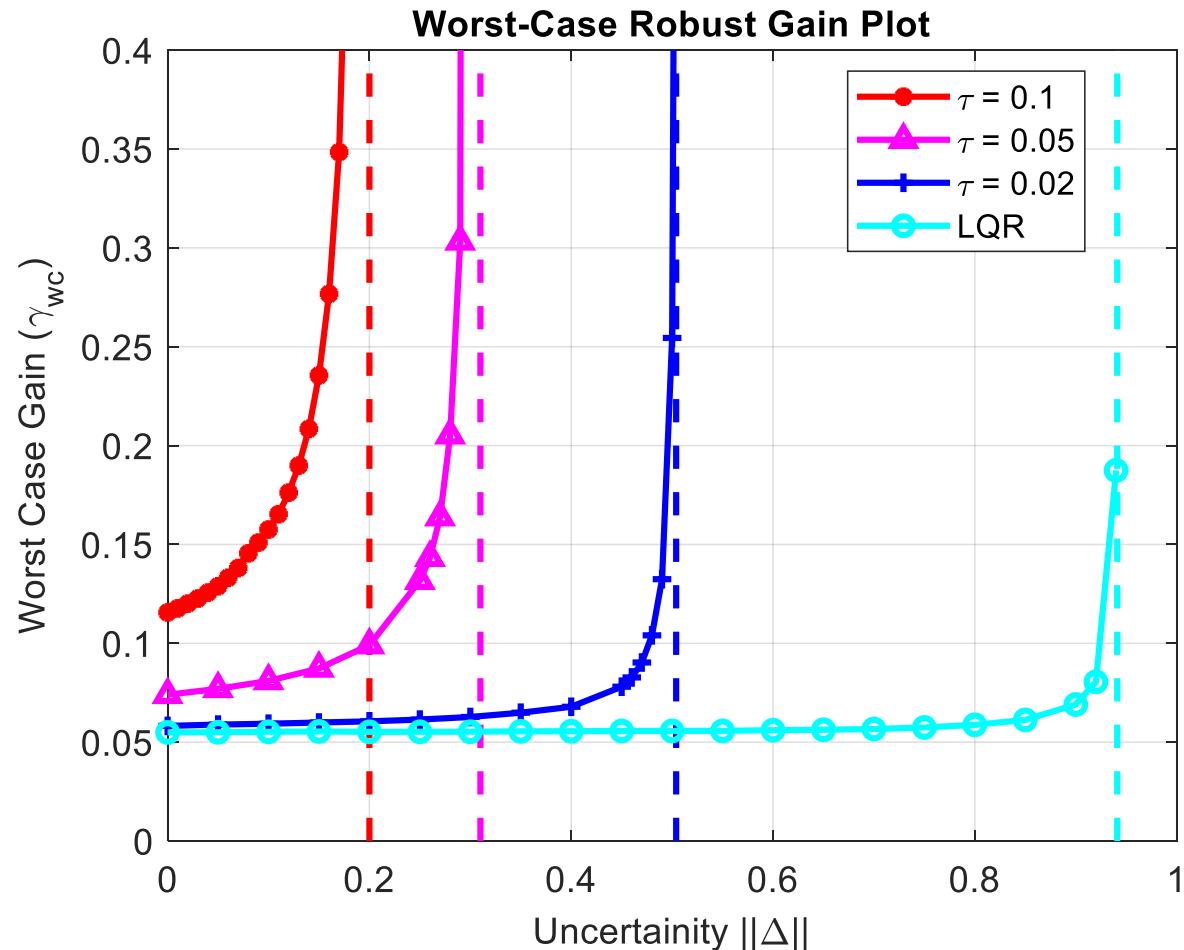
Randomly sample Δ to find “bad” perturbation and compute corresponding worst-case disturbance using method in [1]

[1] Iannelli, Seiler, Marcos, Construction of worst-case disturbances for LTV systems..., 2019.

Closed-Loop Robust L2-to-Euclidean Gain

Two Controllers:

- Finite-Horizon LQR with state feedback
- Output Feedback using high pass filter $\tau s / (\tau s + 1)$ to estimate angular rates

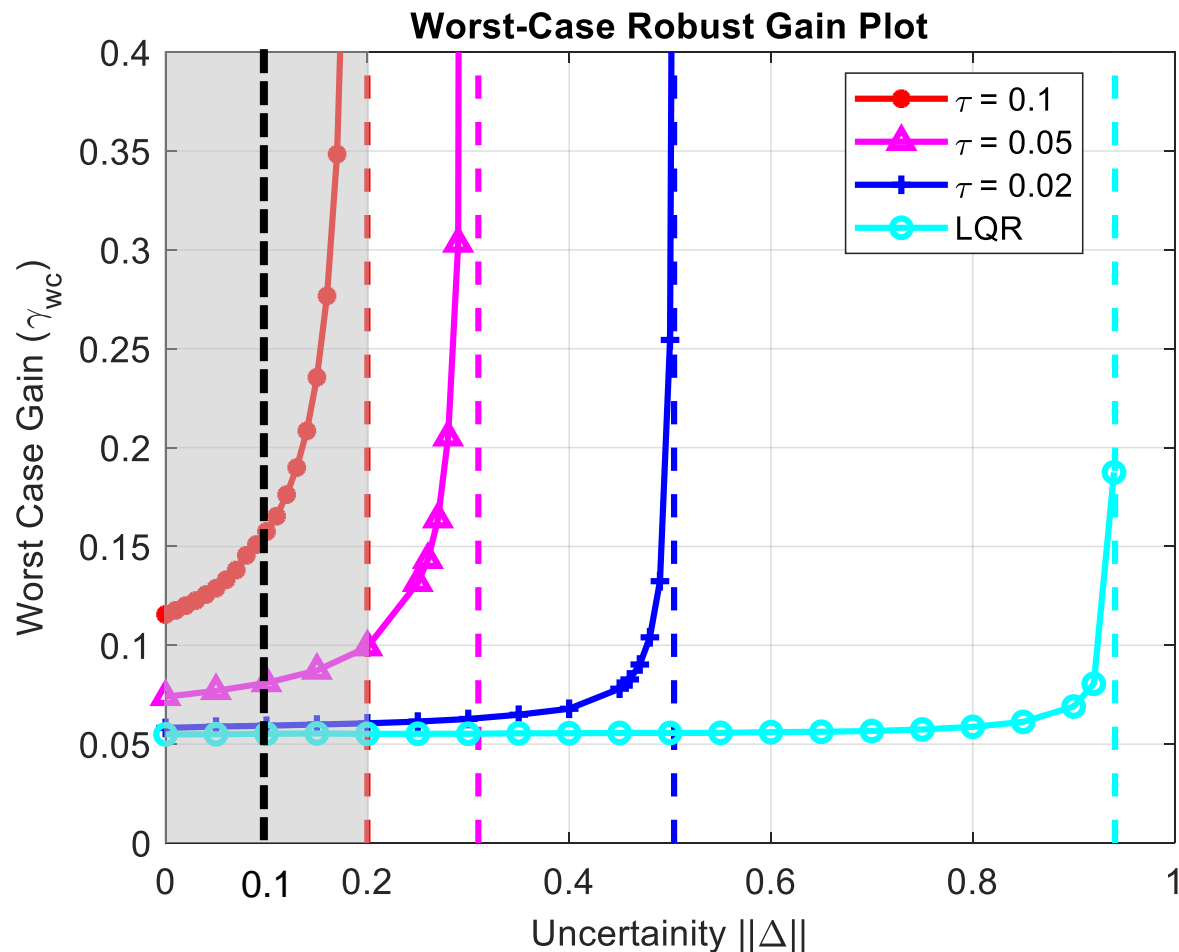


Finite horizon robustness is degraded by output feedback with rate estimates.

Closed-Loop Robust L2-to-Euclidean Gain

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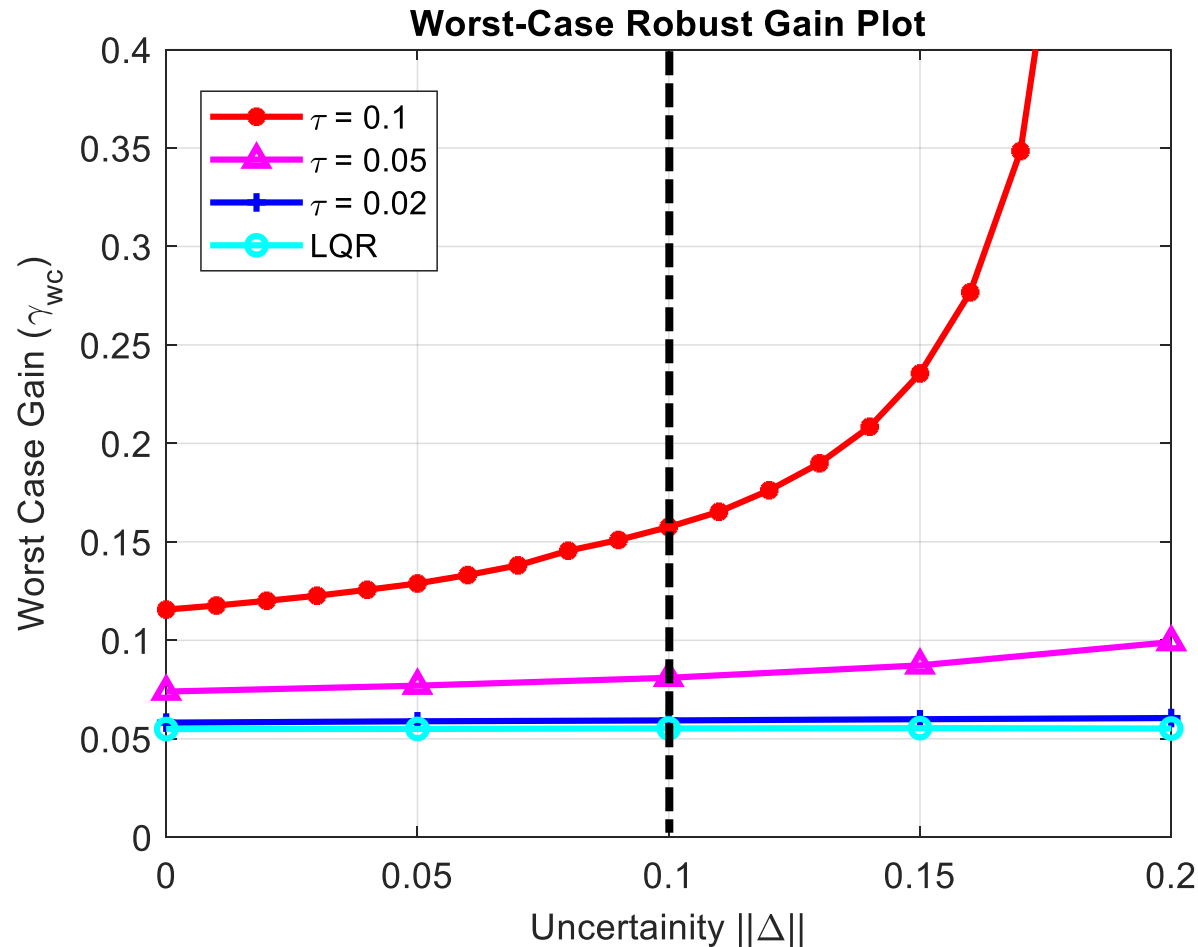


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Closed-Loop Robust L2-to-Euclidean Gain

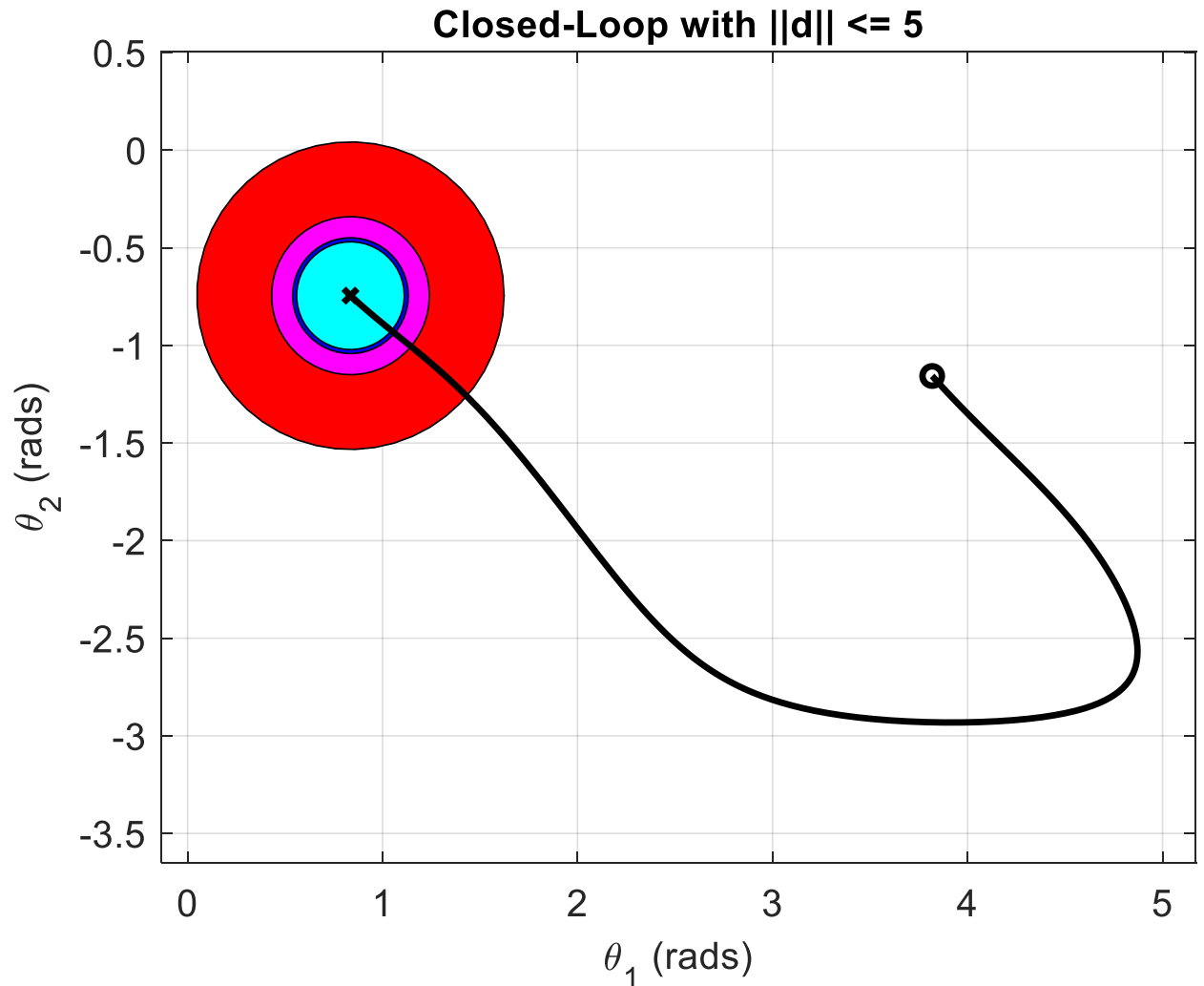
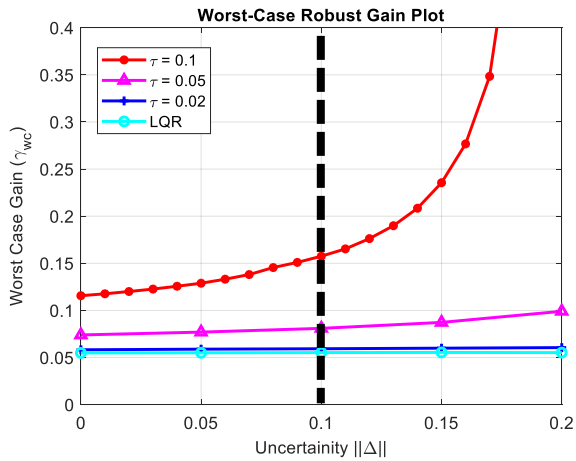
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Finite horizon robustness is degraded by output feedback with rate estimates.

Impact of Using High Pass Rate Estimator

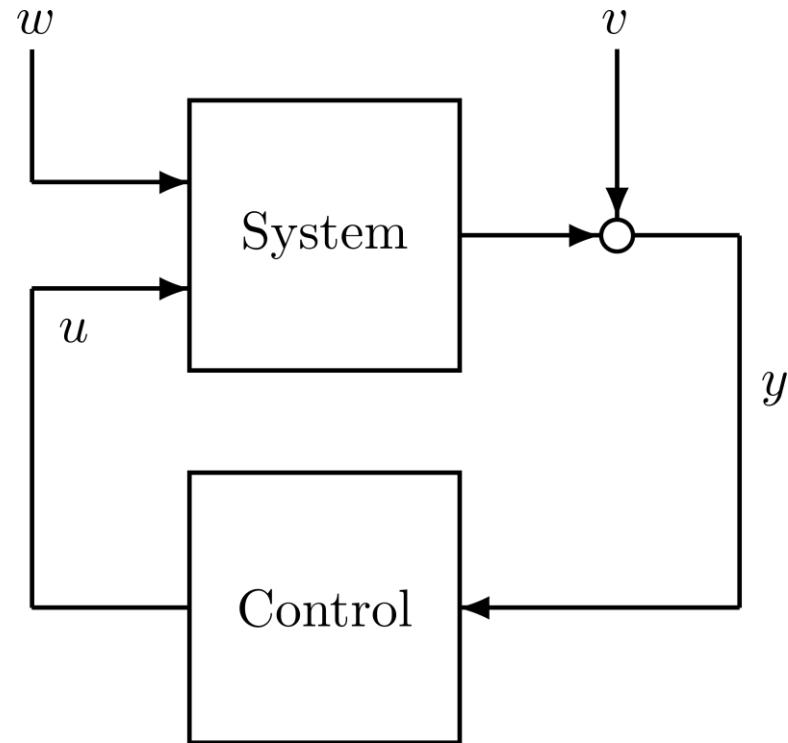


Outline

- Brief Overview of Robust Control
- Robustness of Time-Varying Systems
- **Future Directions**
 - **Robustness in Reinforcement Learning**
 - Design for Optimization Algorithms
- Conclusions

“Model-Free” Reinforcement Learning

- **Goal:** Train a control policy from data to maximize a cumulative reward
 - Training data obtained from a simulator or the real system
 - Often assume state feedback
 - Many algorithms (Q-learning, value iteration, policy iteration, policy search) [1,2,3]
 - Algorithms have close connections to dynamic programming and optimal control.



[1] D.P. Bertsekas, “Reinforcement Learning and Optimal Control,” 2019.

[2] R.S. Sutton and A.G. Barto, “Reinforcement Learning: An Introduction,” 2018.

[3] C. Szepesvári, “Algorithms for Reinforcement Learning,” 2010.

Is Robustness an Issue in RL?

Training via simulation

Training on real system

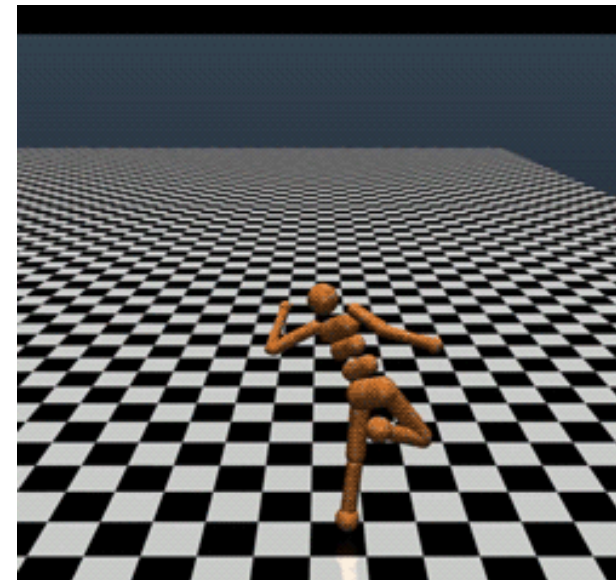
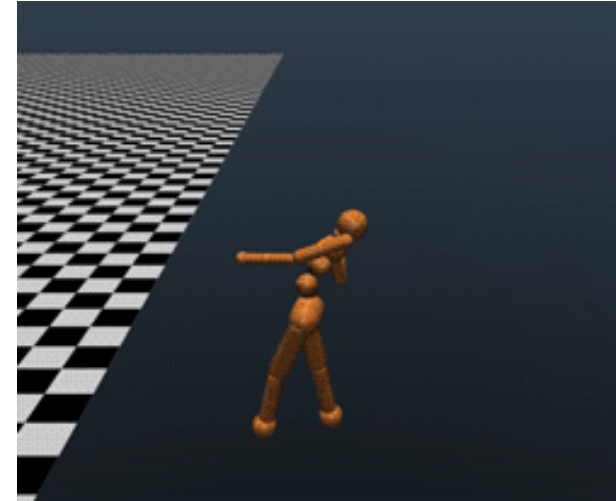
Is Robustness an Issue in RL?

Training via simulation

- Training can exploit flaws in the simulator [1].
- Loss of performance transitioning from simulator to real system.

Training on real system

Robotic Walking in MuJoCo



[1] B. Recht, "A Tour of Reinforcement Learning," arXiv, 2018.

Is Robustness an Issue in RL?

Training via simulation

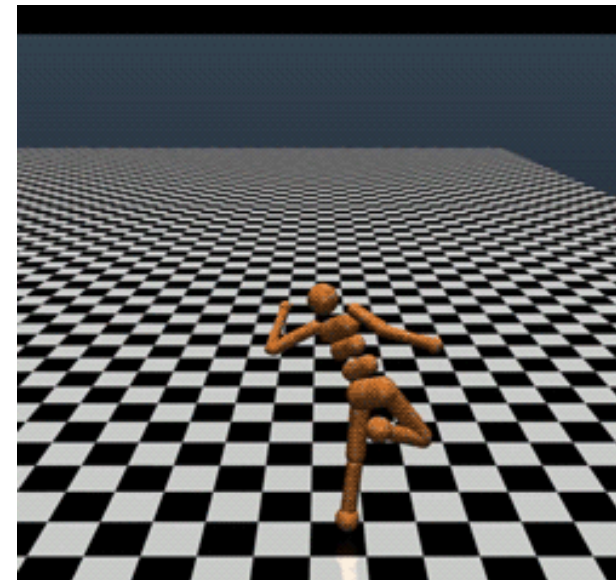
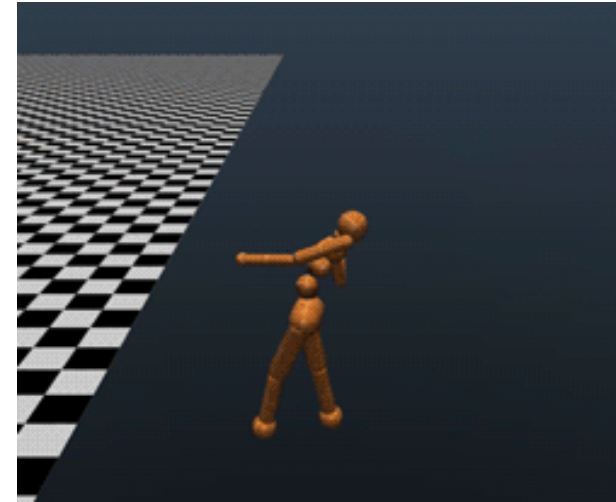
- Training can exploit flaws in the simulator [1].
- Loss of performance transitioning from simulator to real system.

Training on real system

- Part to part variation (train on one system and implement on many)
- Changes in system dynamics over time (temperature dependence, environmental effects, etc....)

[1] B. Recht, "A Tour of Reinforcement Learning," arXiv, 2018.

Robotic Walking in MuJoCo



Summary: Recovering Robustness in RL

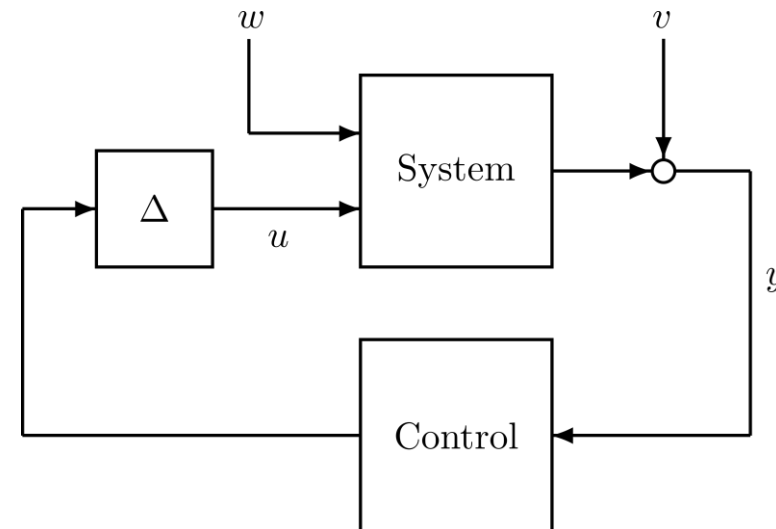
- Robustness issues can arise in output-feedback controllers trained by RL [2]
 - Linear Quadratic Gaussian (LQG) Control can be solved via RL
 - A well-known counterexample by Doyle [1] demonstrates that LQG controllers can have arbitrarily small margins.

[1] J. Doyle. Guaranteed margins for LQG regulators, IEEE TAC, 1978.

[2] Venkataraman & Seiler, Recovering Robustness in Model-Free Reinforcement Learning, '18 arXiv and '19 ACC submission.

Summary: Recovering Robustness in RL

- Robustness issues can arise in output-feedback controllers trained by RL [2]
 - Linear Quadratic Gaussian (LQG) Control can be solved via RL
 - A well-known counterexample by Doyle [1] demonstrates that LQG controllers can have arbitrarily small margins.
- Robustness can be recovered by introducing (synthetic) input perturbations during the RL training [2].



[1] J. Doyle. Guaranteed margins for LQG regulators, IEEE TAC, 1978.

[2] Venkataraman & Seiler, Recovering Robustness in Model-Free Reinforcement Learning, '18 arXiv and '19 ACC submission.

Linear Quadratic Gaussian (LQG)

Minimize $J_{LQG}(u) := \lim_{N \rightarrow \infty} \frac{1}{N} E \left[\sum_{t=0}^N x_t^T Q x_t + u_t^T R u_t \right]$

Subject To: $x_{t+1} = A x_t + B u_t + B_w w_t$
 $y_t = C x_t + v_t$

The optimal controller has an observer/state-feedback form

$$\hat{x}_{t+1} = A \hat{x}_t + B u_t + L (y_t - C \hat{x}_t)$$
$$u_t = -K \hat{x}_t$$

Gains (K, L) computed by solving two Riccati equations.

This solution is model-based, i.e. it uses data A, B, C , etc

Reinforcement Learning

- Partially Observable Markov Decision Processes (POMDPs)
 - Set of states, S
 - Set of actions, A
 - Reward function, $r: S \times A \rightarrow \mathbb{R}$
 - State transition probability, T
 - Set of observations and observation probability, O
- Many methods to synthesize a control policy from input/output data to maximize the cumulative reward

$$J_{RL}(a) := E \left[\sum_{t=0}^N r(s_t, a_t) \right]$$

- The LQG problem is a special case of this RL formulation

Doyle's Example ('78 TAC)

- LQR state-feedback regulators have provably good margins.
- Doyle's example shows that LQG regulators can have arbitrarily small input margins.

Honeywell Interoffice Correspondence

Date: August 23, 1977
To: C. A. Harvey
From: J. C. Doyle
Location: S&RC, Research



cc: L. Q. Gaussian
J. A. Hauge
A. P. Kizilos
A. F. Konar
E. E. Yore
N. R. Zagalsky
Systems and Control Technology

Subject: "Guaranteed Margins for LQG Regulators"

A B S T R A C T

There aren't any.

All engineers who have been using LQG methodology may pick up their Nichols charts from the supply room.

Doyle's Example ('78 TAC)

- LQR state-feedback regulators have provably good margins.
- Doyle's example shows that LQG regulators can have arbitrarily small input margins.
- Doyle's example can also be solved within RL framework using direct policy search:

$$\begin{aligned}z_{t+1} &= A_K(\theta)z_t + B_K(\theta)y_t \\ u_t &= C_K(\theta)z_t\end{aligned}$$

where

$$A_K(\theta) := \begin{bmatrix} 0 & \theta_1 \\ 1 & \theta_2 \end{bmatrix}, B_K(\theta) := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C_K^T(\theta) := \begin{bmatrix} \theta_3 \\ \theta_4 \end{bmatrix}$$

- RL will converge to the optimal LQG control with infinite data collection. Thus RL can also have poor margins.

Recovering Robustness

- Increase process noise during training?
 - This causes margins to decrease on Doyle's example
 - Process noise is not model uncertainty

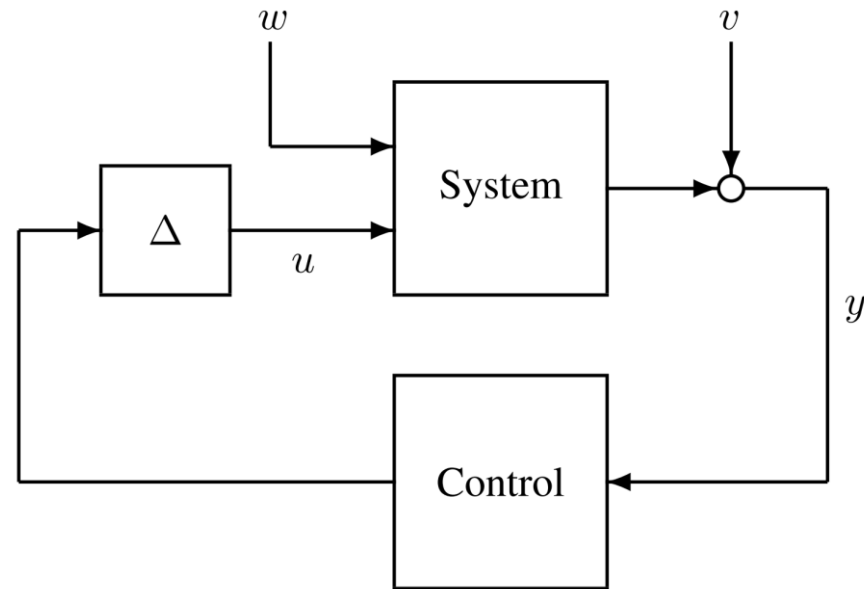
Recovering Robustness

- Increase process noise during training?
- Modify reward to increase state penalty or decrease control penalty?
 - Again, this causes margins to decrease on Doyle's example
 - Trading performance vs. robustness via the reward function can be difficult or counter-intuitive

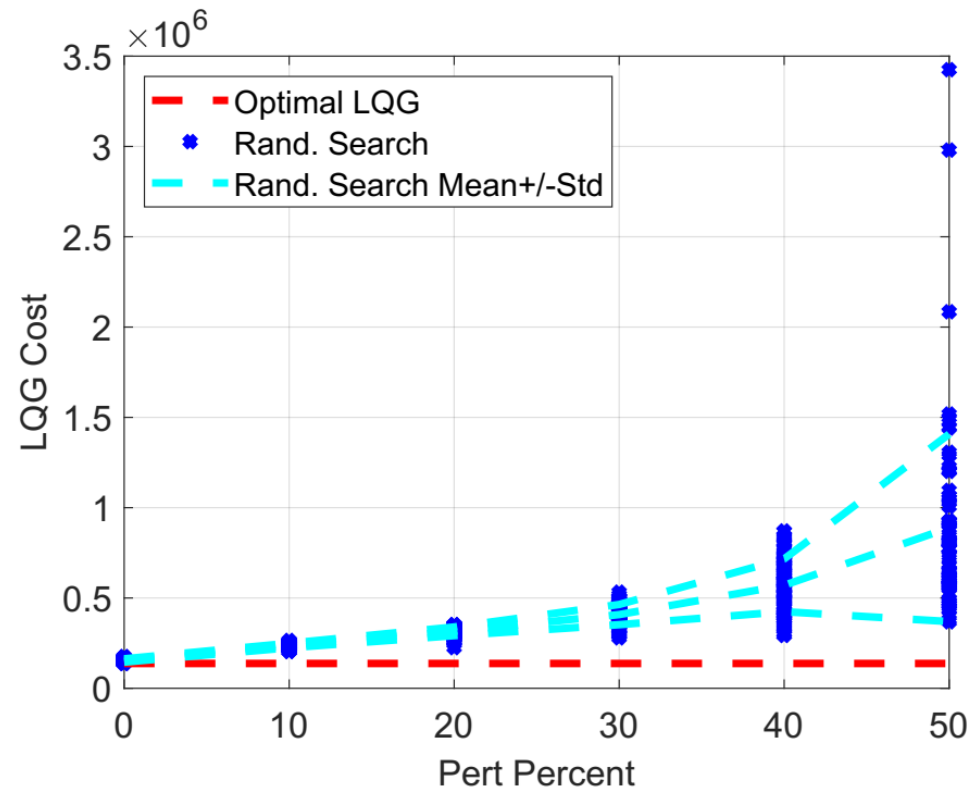
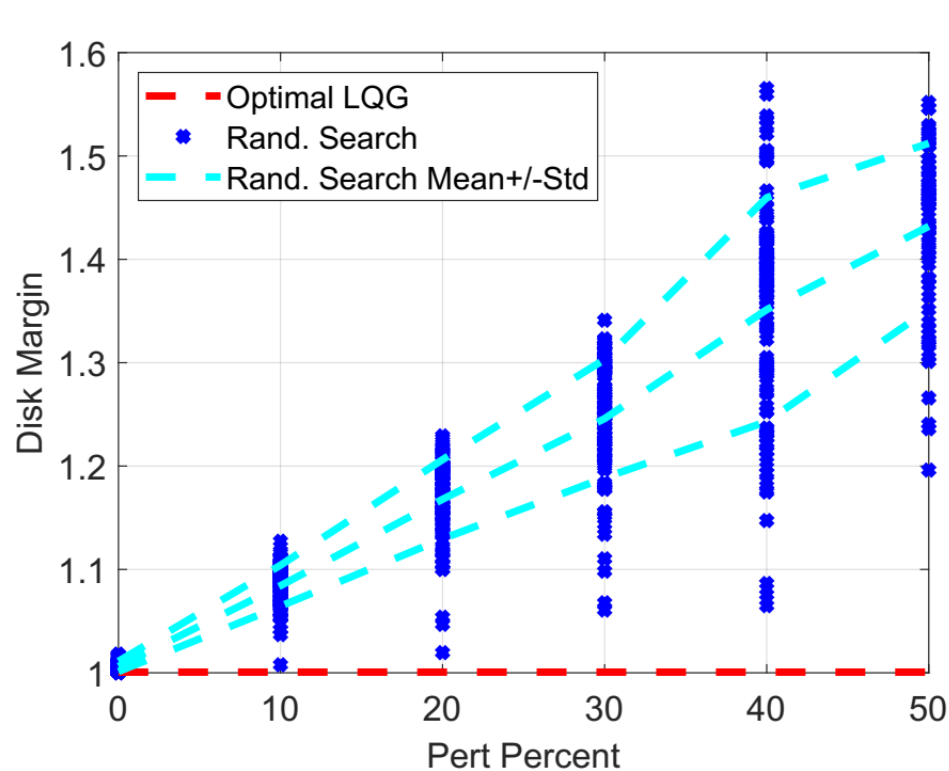
Recovering Robustness

- Increase process noise during training?
- Modify reward to increase state penalty or decrease control penalty?
- Inject synthetic gain/phase variations at the plant input (and output?) during the training phase.

$\Delta = 1 + \delta$ where
 δ is $U[-b, b]$

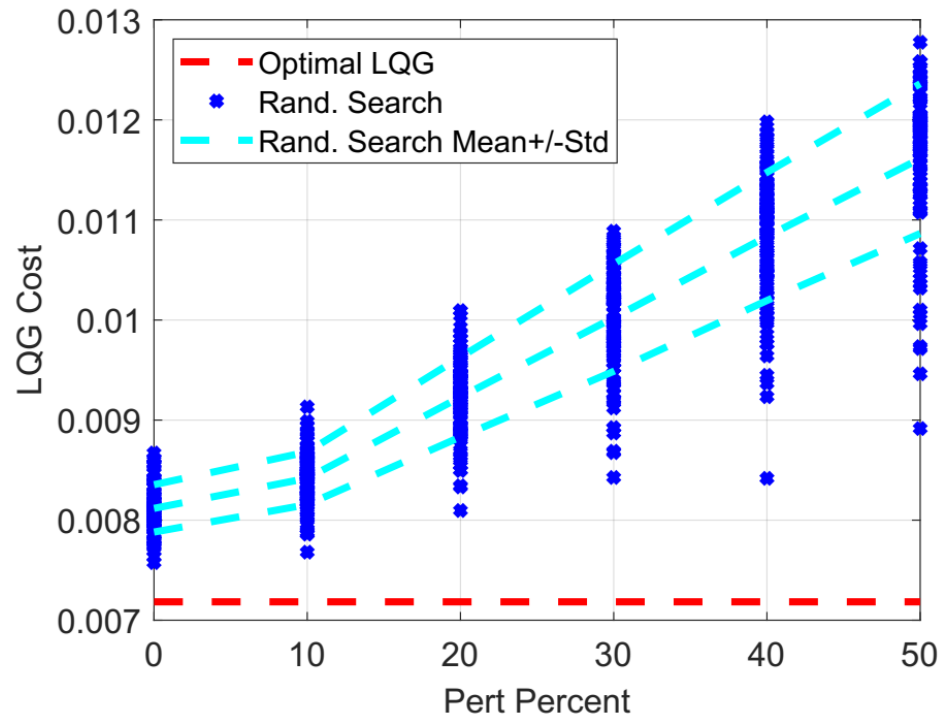
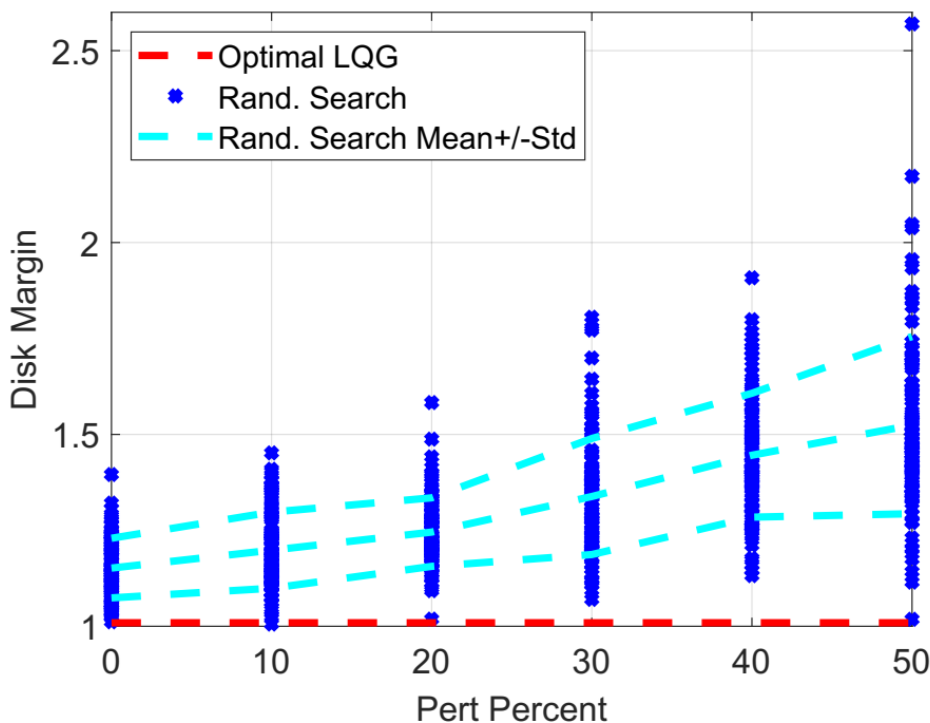


Results On Doyle's Example



Results on Simplified Flexible System

- Model has 4-states (Rigid body and lightly damped modes)
- RL applied to 3-state controller parameterization
 - LQG controller is not in the control policy parameterization
 - Still converges to policy with small margins
 - Robustness recovered with synthetic perturbations during training



Next Steps

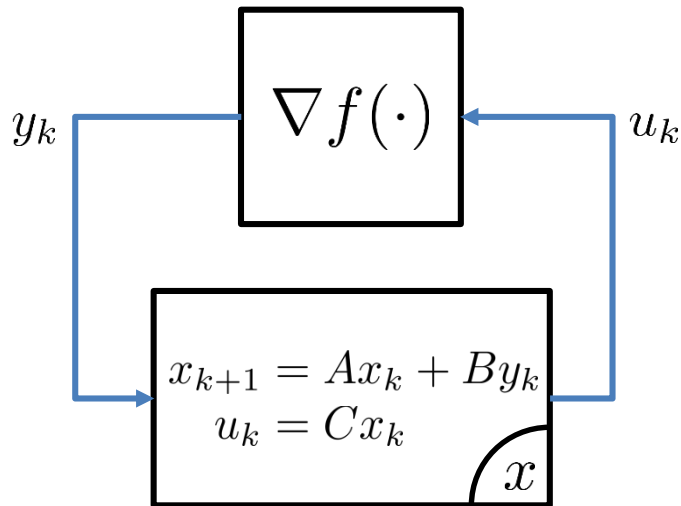
- How should synthetic perturbations be introduced during training?
- Can we make any rigorous claims about the proposed method? Performance certification?
- Impact of fundamental performance limits on RL policies?

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First-order optimization algorithms as robust control

$$\min_{x \in \mathbf{R}^n} f(x)$$



*Lessard, Hu, Recht,
Seiler, Rantzer, Packard*

Assumptions on f (uncertain plant)

- Strongly convex (m)
- Lipschitz gradients (L)

Algorithm (controller)

- Finite-dim, strictly proper, linear system
- input: gradient at iterate
- output: next iterate

Automated Analysis with IQC/SDP

- characterize ∇f with IQCs
- certify convergence-rate of interconnection

Extensions

- Gradient noise
- Constrained optimization
- Algorithm design
- Stochastic Gradient Descent

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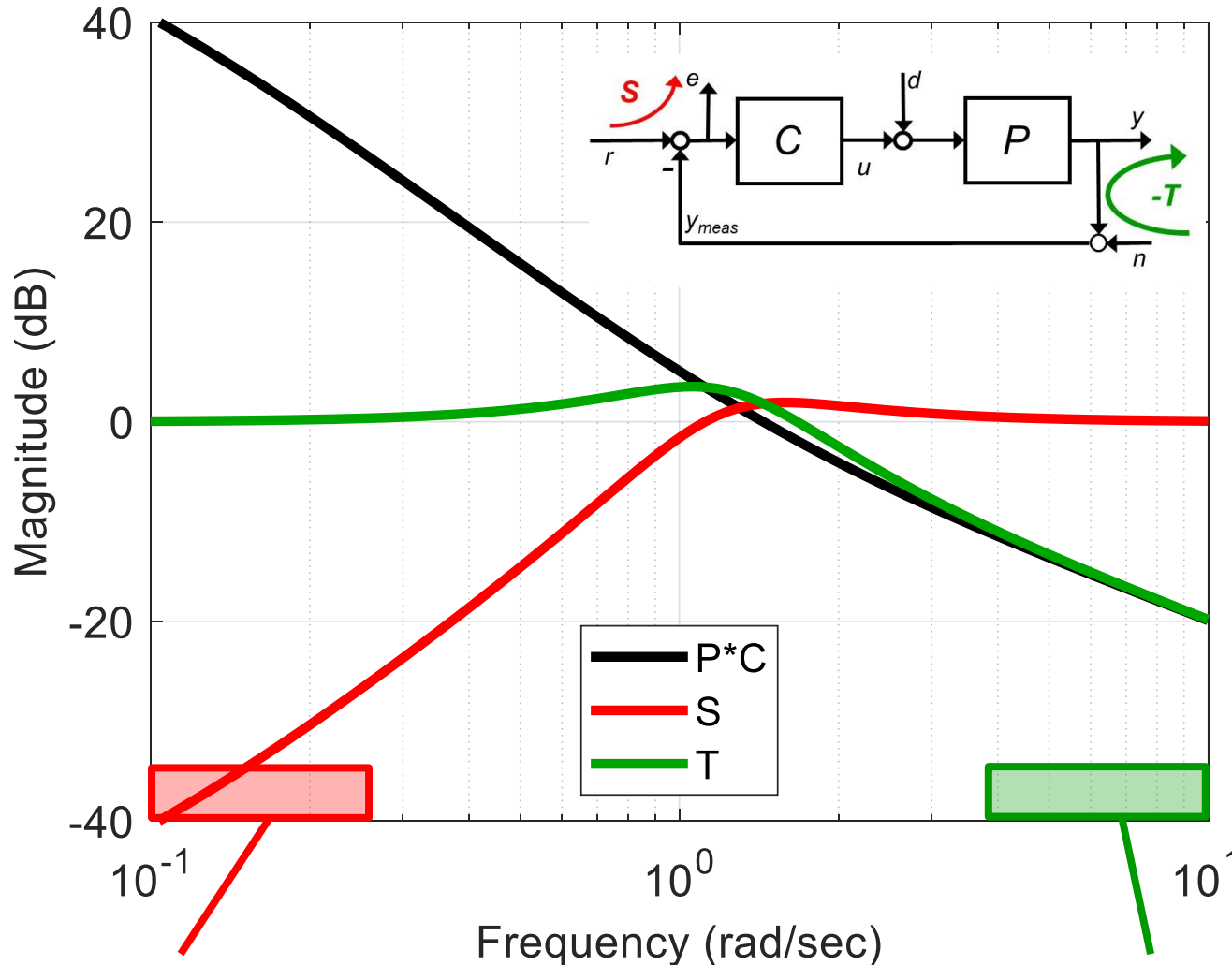
Conclusions

- Robust control has a long history with many successes
 1. Multivariable Optimal Control
 2. Fundamental Limitations of Control
 3. Uncertainty Modeling and Robustness Analysis
- Robust control techniques can solve emerging problems
 1. Robustness in controls designed via data-driven (RL) methods
 2. Design and analysis of optimization algorithms
- Acknowledgements:
 - Funding: NSF, AFOSR, ONR, NASA, Seagate, MSI, Xcel RDF, MnDrive
 - Past PhDs & Visitors: Annoni, Hu, Honda, Kotikalpudi, Lacerda, Ossmann, Peni, Pfifer, Takarics, Theis, Venkataraman, Wang

<https://www.aem.umn.edu/~SeilerControl/>

Backup Slides

Typical S+T=1 Tradeoff



$$P = \frac{1}{s+0.1}$$

$$C = \frac{s+1.5}{s}$$

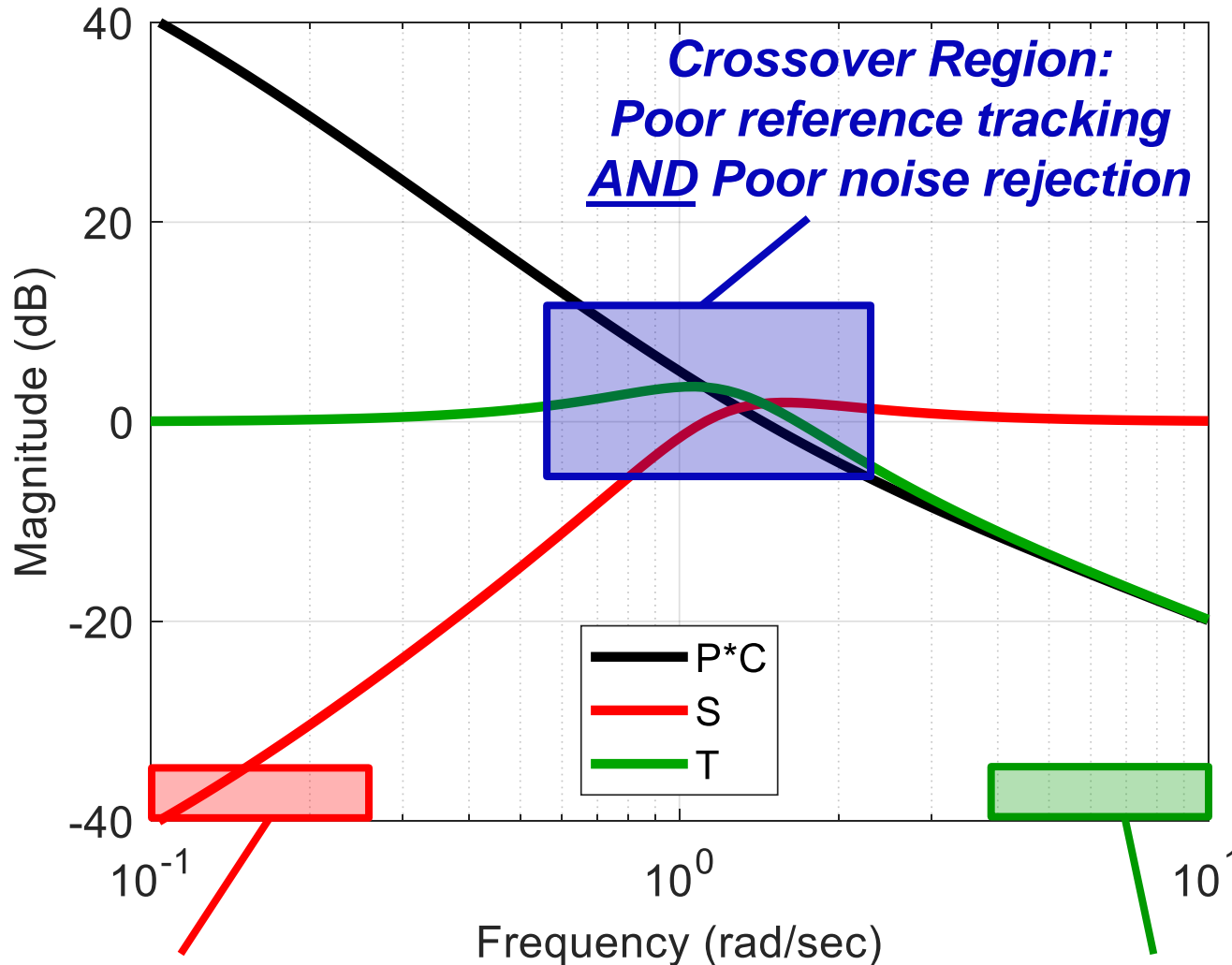
$$S = \frac{1}{1+PC}$$

$$T = \frac{PC}{1+PC}$$

Large loop gain $|PC|$:
Good reference tracking
Poor noise rejection

Small loop gain $|PC|$:
Poor reference tracking
Good noise rejection

Typical S+T=1 Tradeoff



$$P = \frac{1}{s+0.1}$$

$$C = \frac{s+1.5}{s}$$

$$S = \frac{1}{1+PC}$$

$$T = \frac{PC}{1+PC}$$

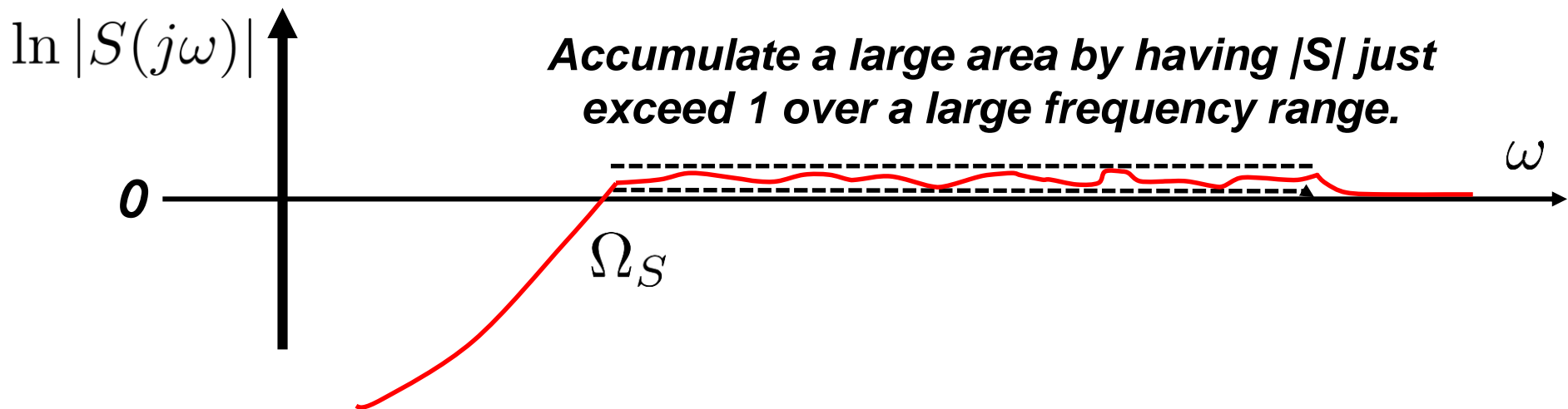
Large loop gain $|PC|$:
Good reference tracking
Poor noise rejection

Small loop gain $|PC|$:
Poor reference tracking
Good noise rejection

Bode Integral Theorem and “Peaking”

A procedure to avoid peaking could be:

- Obtain significant Sensitivity reduction over $[0, \Omega_S]$.
This incurs a large negative integral which must be balanced.
- Maintain $|S(j\omega)|$ slightly larger than 1 over a wide interval.
This incurs a positive integral balancing the negative integral.
- Make $|PC|$ approach 0 quickly at higher frequencies so that $|S|$ quickly approaches 1.

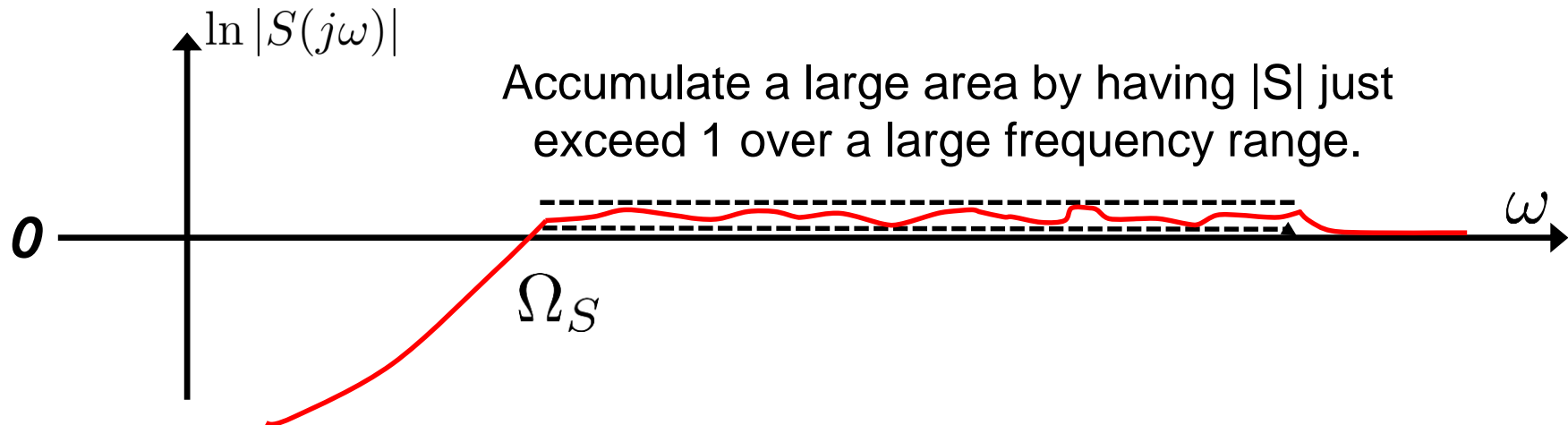


Available Bandwidth

The Bode Integral theorem may appear to be a minor constraint, e.g. spreading area over a large frequency band.

Stein ('89 Bode Lecture, '03 CSM):

a key fact about physical systems is that they do not exhibit good frequency response fidelity beyond a certain bandwidth. ... Let us call that bandwidth the “available bandwidth,” Ω_a



Available Bandwidth

The Bode Integral theorem may appear to be a minor constraint, e.g. spreading area over a large frequency band.

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a key fact about physical systems is that they do not exhibit good frequency response fidelity beyond a certain bandwidth. ... Let us call that bandwidth the “available bandwidth,” Ω_a

The available bandwidth due to physical (hardware) constraints requires positive area be accumulated over a finite frequency band.

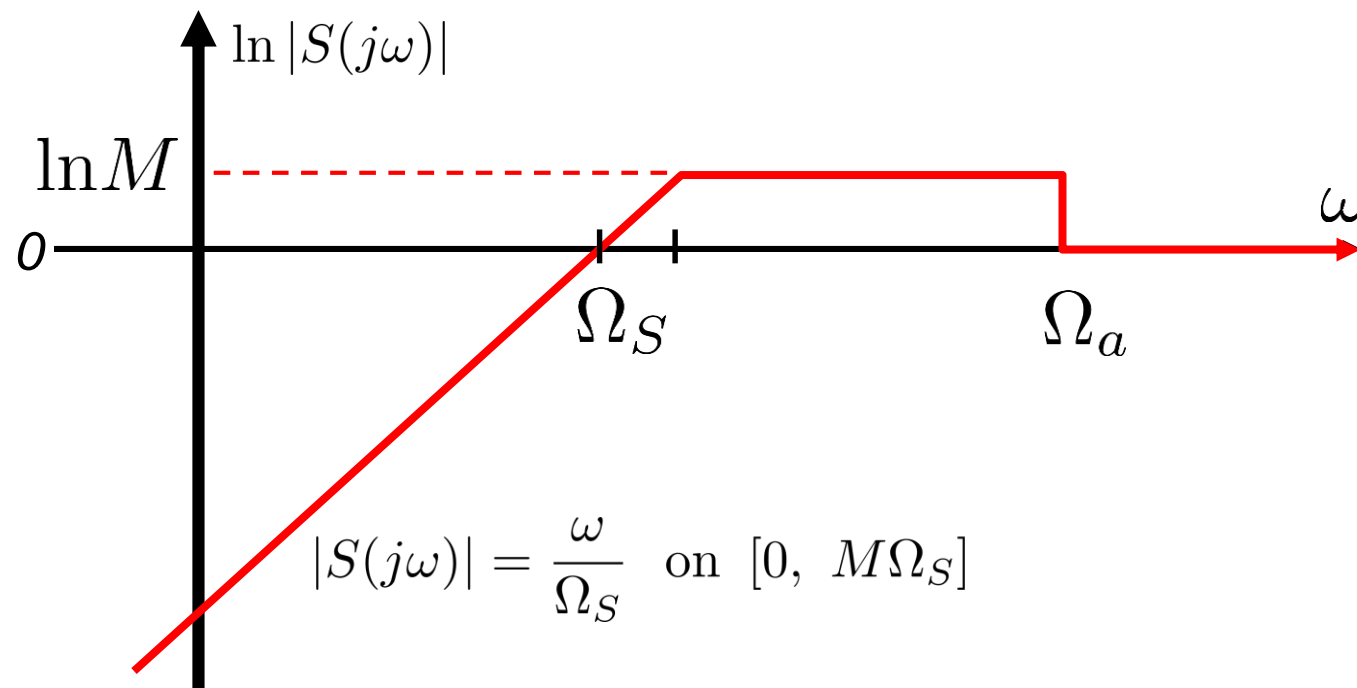
Consequence: Improving performance (e.g. increased bandwidth) comes at the expense of reduced robustness (peak in $|S|$).

Consequence of Available Bandwidth

$|PC|$ must roll-off quickly above Ω_a

$$\int_0^{\infty} \ln |S(j\omega)| d\omega = \pi p \quad \xrightarrow{\text{roughly}} \quad \Omega_S \leq \frac{\Omega_a \ln M - \pi p}{M}$$

Performance is constrained by the Bode integral and robustness requirements.



Partial Dictionary of IQCs [1]

Uncertainty

1. Passive

2. Norm-bounded LTI

3. Constant Real Parameter

4. Varying Real Parameter

5. Unit Saturation

IQC Multiplier

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

$$\begin{bmatrix} X(j\omega) & 0 \\ 0 & -X(j\omega) \end{bmatrix} \text{ where } X(j\omega) \geq 0$$

$$\begin{bmatrix} X(j\omega) & Y(j\omega) \\ Y(j\omega)^* & -X(j\omega) \end{bmatrix} \text{ where } X(j\omega) \geq 0 \text{ and } Y(j\omega) = -Y(j\omega)^*.$$

$$\begin{bmatrix} X & Y \\ Y^T & -X \end{bmatrix} \text{ where } X \geq 0 \text{ and } Y = -Y^T.$$

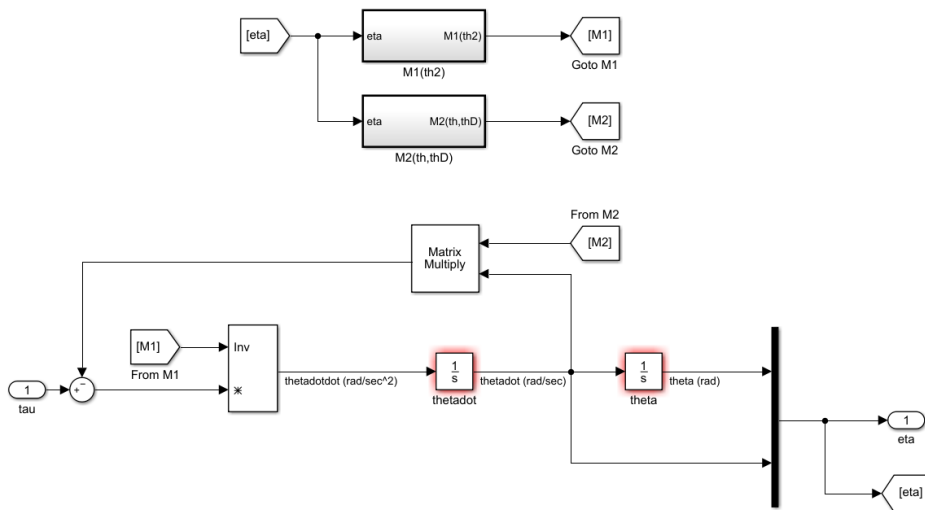
$$\begin{bmatrix} 0 & 1 + H(j\omega) \\ 1 + H(j\omega)^* & -2(1 + \text{Re}H(j\omega)) \end{bmatrix} \text{ where } \|h\|_1 \leq 1.$$

[1] Megretski & Rantzer, System analysis via IQCs, TAC, 1997. [IQCs derived based on much prior literature]

LTV Toolchain

```
% Matlab snapshot linearizations
% along nominal trajectory
io(1) = linio('TwoLinkRobotOL/Input Torque',1,'input');
io(2) = linio('TwoLinkRobotOL/Two Link Robot Arm',1,'output');
sys = linearize('TwoLinkRobotOL',io,Tgrid);
```

```
% Construction of LTV Model
G = tvss(sys,Tgrid);
```



Simulink Model of Robotic Arm