Robust Analysis and Synthesis for Linear Parameter Varying Systems

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Lockheed Martin BFF
Gary Balas (1960-2014)
Outline

1. Flexible Aircraft
2. Linear Parameter Varying Systems
3. Robustness Analysis
4. Summary
Outline

1 Flexible Aircraft

2 Linear Parameter Varying Systems

3 Robustness Analysis

4 Summary
Efficient aircraft design

- lightweight structures
- high aspect ratios
Why Flexible Wings?

Breguet Range Equation

\[
\text{Range} = V \times I_{sp} \times \frac{\text{Lift}}{\text{Drag}} \times \ln\left(\frac{m_{\text{takeoff}}}{m_{\text{landing}}}\right)
\]

- \(V\): Propulsion efficiency
- \(I_{sp}\): Glide number
- \(\text{Lift}/\text{Drag}\): Structural mass
Why Flexible Wings?

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\]

Induced Drag for elliptic (optimal) lift distribution:

\[
\text{Induced Drag} = \frac{\text{Lift}^2}{\pi \Lambda}
\]

\( \Rightarrow \) Maximize wing aspect ratio \( \Lambda \)
Why Flexible Wings?

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\[\Rightarrow \text{Maximize wing aspect ratio } \Lambda\]

Main contributions to total mass:

\[
m_{\text{takeoff}} = m_{\text{structure}} + m_{\text{payload}} + m_{\text{fuel}}
\]

\[
m_{\text{landing}} = m_{\text{structure}} + m_{\text{payload}}
\]

\[\Rightarrow \text{Minimize structural mass } m_{\text{structure}}\]
Why Flexible Wings?

**Breguet Range Equation**

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  \[ m_{\text{landing}} = m_{\text{structure}} + m_{\text{payload}} \]
  \[ \Rightarrow \text{Minimize structural mass } m_{\text{structure}} \]

**Light weight, high aspect ratio, flexible wings**
Classical Approach

Rigid Body Modes

Frequency
Classical Approach

- Rigid Body Modes
- Aeroelastic Modes

Frequency
Classical Approach

- Rigid Body Modes
- Aeroelastic Modes

Frequency Separation

Frequency
Classical Approach

Controller Bandwidth

Rigid Body Modes

Aeroelastic Modes

Frequency Separation

Flight Dynamics, Classical Flight Control
Classical Approach

Controller Bandwidth

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Flight Dynamics, Classical Flight Control

Flutter Analysis

Frequency
Flexible Aircraft Challenges

- Rigid Body Modes
- Aeroelastic Modes

Frequency
Flexible Aircraft Challenges

- Rigid Body Modes
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Coupling between Rigid Body and Aeroelastic Modes, Body Freedom Flutter

Integrated Control Design
Flexible Aircraft Challenges

Coupling between Rigid Body and Aeroelastic Modes, Body Freedom Flutter
Coupling between Rigid Body and Aeroelastic Modes, Body Freedom Flutter
Body Freedom Flutter
**Flight Dynamics**

- Inertial Forces
- Aerodynamic Forces

**Rigid Body Dynamics**

- Classical 6 degree of freedom equations of motion
- Steady aerodynamics
Aeroservoelastic Model

Aeroelasticity

Flexible Aircraft

- Rigid body dynamics (6 DoF)
- Structural dynamics (typically 6-8 modes)
- Unsteady aerodynamics (typically 2 lag states per mode)
Aeroservoelasticity

High dimensional, strongly coupled models

- Rigid body dynamics (from flight dynamics)
- Structural dynamics (from finite element method)
- Unsteady aerodynamics (from potential theory)
Body Freedom Flutter Vehicle
Aerospace Engineering and Mechanics

mini-MUTT Aircraft at UMN

Key Features:

- Low-cost, modular flight research infrastructure
- Design based on the Lockheed Martin BFF vehicle
- Parallels X-56 Flight test program at NASA
- Fabricated completely in-house
- Detachable wings of various flexibility
Flight Test of Rigid Wing mini-MUTT
Next Steps:

- Finish building flexible wings
- Flight test campaign this summer
Limitation of Classical Approaches

Classical approaches are not suitable for control of flexible aircraft

Parameter Dependent Dynamics

Model Uncertainty

Aerodynamics:
- Simple potential theory based model
- Rational approximation of unsteady effects

Structural Dynamics:
- Simple beam model
- Estimates of mass and inertia properties
1 Flexible Aircraft

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Nonlinear equation of motion:

\[ \dot{x}(t) = f(x(t), u(t), \rho(t)) \]
\[ y(t) = h(x(t), u(t), \rho(t)), \]

where \( \rho \) is a vector of measurable, exogenous signals, in this case airspeed.

**Parameterized Trim Points:** Assume there are trim points \((\bar{x}(\rho), \bar{u}(\rho), \bar{y}(\rho))\) parameterized by \( \rho \):

\[ 0 = f(\bar{x}(\rho), \bar{u}(\rho), \rho) \]
\[ \bar{y}(\rho) = h(\bar{x}(\rho), \bar{u}(\rho), \rho) \]
Nonlinear equation of motion:

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\end{align*}
\]

where \( \rho \) is a vector of measurable, exogenous signals, in this case airspeed.

Time-Varying Linearization: Linearize around \((\bar{x}(\rho(t)), \bar{u}(\rho(t)), \bar{y}(\rho(t)), \rho(t))\)

\[
\begin{align*}
\dot{\delta}_x &= A(\rho)\delta_x + B(\rho)\delta_u + \Delta f(\delta_x, \delta_u, \rho) - \dot{x}(\rho) \\
\dot{\delta}_y &= C(\rho)\delta_x + D(\rho)\delta_u + \Delta h(\delta_x, \delta_u, \rho)
\end{align*}
\]

where \( A(\rho) := \frac{\partial f}{\partial x} (\bar{x}(\rho), \bar{u}(\rho), \rho), \) etc.
\[ \dot{x}(t) = A(\rho(t))x(t) + B(\rho(t))u(t) \]
\[ y(t) = C(\rho(t))x(t) + D(\rho(t))u(t) \]

Parameter vector $\rho$ lies within a set of admissible trajectories

\[ \mathcal{A} := \{ \rho : \mathbb{R}^+ \to \mathbb{R}^{n_\rho} : \rho(t) \in \mathcal{P}, \dot{\rho}(t) \in \dot{\mathcal{P}} \ \forall t \geq 0 \} \]

Comments:

- LPV theory is an extension of classical gain-scheduling used in industry, e.g. flight controls.
- Large body of literature in 90s: Shamma, Packard, Gahinet, Scherer, and many others.
- LPVTools: Toolbox developed by Balas, Packard, Seiler, and Hjartarson.
\[
\begin{align*}
\dot{x}(t) &= A(\rho(t))x(t) + B(\rho(t))u(t) \\
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**Grid based LPV systems**

**LFT based LPV systems**
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Integral Quadratic Constraints (IQC)

IQC provide a general framework for analysis of a known LTI system $G$ under perturbations $\Delta$ (Megretski & Rantzer, '97 TAC).

**Goal:** Extend framework to cases where known system is LPV, e.g. robustness margins for flexible aircraft.
Example: Passive System

\[ w = \Delta(v, t) \text{ is a passive system (pointwise in time).} \]

\[ 2v(t)^T w(t) \geq 0 \ \forall t \]
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\[ 2v(t)^T w(t) \geq 0 \quad \forall t \]

\[ \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} \geq 0 \quad \forall t \]

**Pointwise quadratic constraint**
Theorem: Assume:

1. Interconnection is well-posed.
2. $\Delta$ is (pointwise) passive.
3. $\exists V \geq 0$ such that

$$
\dot{V} + \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} < d^T d - e^T e
$$

Then gain from $d$ to $e$ is $\leq 1$. 
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\]

Then gain from $d$ to $e$ is $\leq 1$.

Proof: Let $d \in L[0, \infty)$ be any input signal and $x(0) = 0$. Integrate:
\[
V(x(T)) + \int_0^T \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} dt < \int_0^T d(t)^T d(t) dt - e(t)^T e(t) dt
\]

Left side is $\geq 0$ by $V \geq 0$ and passivity.
**Theorem:** Assume:

1. Interconnection is well-posed.
2. $\Delta$ is (pointwise) passive.
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\dot{V} + \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} < d^T d - e^T e
\]

Then gain from $d$ to $e$ is $\leq 1$.

**Comments:**
1. The proof relied on $V \geq 0$ and the passivity constraint. More general integral quadratic constraints (IQC) can be incorporated, e.g. Zames-Falb.

2. Eq (1) is a matrix inequality when $G$ is LTI and $V$ is quadratic. Convex optimization can be used to efficiently search over combinations of IQCs.
**Time Domain:**

Let $\Psi$ be a stable, LTI system and $M$ a constant matrix. $\Delta$ satisfies IQC defined by $\Psi$ and $M$ if

$$\int_0^T z(t)^T M z(t) dt \geq 0$$

$\forall v \in L_2[0, \infty)$, $w = \Delta(v)$, and $T \geq 0$. 

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**Diagram:**

- Input $v$ to $\Delta$ produces $w$.
- $\Delta$ also outputs $w$.
- $\Psi$ processes $z$.
- The system diagram includes feedback paths for $v$ and $w$. 

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**General IQCs (Megretski/Rantzer, ’97 TAC)**
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**Frequency Domain:**
Let $\Pi : j\mathbb{R} \rightarrow \mathbb{C}^{m \times m}$ be Hermitian-valued. $\Delta$ satisfies IQC defined by $\Pi$ if
\[
\int_{-\infty}^{\infty} \left[ \hat{v}(j\omega) \right]^* \Pi(j\omega) \left[ \hat{w}(j\omega) \right] d\omega \geq 0
\]
$\forall v \in L_2[0, \infty)$ and $w = \Delta(v).$
Time Domain:
Let $\Psi$ be a stable, LTI system and $M$ a constant matrix. $\Delta$ satisfies IQC defined by $\Psi$ and $M$ if
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Frequency Domain:
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\]
$\forall v \in L_2[0, \infty)$ and $w = \Delta(v)$.

A non-unique factorization $\Pi = \Psi^\sim M \Psi$ connects the two definitions.
Summary:

1. Analysis involves frequency domain conditions on \( G \) and IQC multiplier(s) \( \Pi \).
2. Proof uses a homotopy method.
3. Any stable factorization \( \Pi = \Psi \sim M \Psi \) and KYP lemma leads to an LMI.
4. LMI condition can be written as:

\[
\dot{V} + z^T M z < d^T d - e^T e
\]

Neither \( V \geq 0 \) nor \( \int_0^T z(t)^T M z(t) \, dt \geq 0 \) holds, in general.

Question:
Is there an equivalent dissipation inequality proof?
Summary:
Under some technical conditions, the frequency-domain conditions in (M/R, '97 TAC) are equivalent to the time-domain dissipation inequality conditions.
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Def.: \( \Pi = \Psi \sim M \Psi \) is a J-Spectral factorization if \( M = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \) and \( \Psi, \Psi^{-1} \) are stable.

Thm.: If \( \Pi = \Psi \sim M \Psi \) is a J-spectral factorization then:
1. \( \Delta \) satisfies the freq. domain IQC (\( \Pi \)) iff it satisfies the time domain IQC (\( \Psi, M \)).
2. All solutions of KYP LMI satisfy \( P \geq 0 \).

Proof: Uses LQ dynamic games, (Willems. ’72 TAC) and (Engwerda, ’05).
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Thm.: Partition \( \Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12}^* \\ \Pi_{21} & \Pi_{22} \end{bmatrix} \). \( \Pi \) has a J-spectral factorization if \( \Pi_{11}(j\omega) > 0 \) and \( \Pi_{22}(j\omega) < 0 \) \( \forall \omega \in \mathbb{R} \cup \{+\infty\} \).

Proof: Use equalizing vectors thm. of Meinsma (SCL, 1995) ■.
Summary:
Under some technical conditions, the frequency-domain conditions in (M/R, '97 TAC) are equivalent to the time-domain dissipation inequality conditions.

Consequences:
The time-domain dissipation inequality conditions can be extended for:

1. LPV robustness analysis (Pfifer & Seiler, '14 IJRNC); (Pfifer & Seiler, in prep.)
2. LPV robust synthesis for general case (Wang, Pfifer, & Seiler, submitted to Aut) and robust filter/feedforward synthesis (Venkataraman & Seiler, in prep.)
3. Optimization analysis with $\rho$-hard IQCs (Lessard, Recht, & Packard)
4. Nonlinear analysis using SOS techniques

Item 1 has been implemented in LPVTools. Item 2 parallels results by (Scherer, Kose, and Veenman) for LFT-type LPV systems.
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Upcoming Flight Test Plans

**NASA X-56a:**
- A. Hjartarson (Musyn) used LPVTools to synthesis (nominal) LPV controllers and assess robustness.
- NASA designed their own gain-scheduled control law.

**UMN mini-MUTT:**
- Finish flex wing and begin flight tests.
- Validate control-oriented aeroelastic models incorporating data from flight tests and high fidelity CFD/CSD models.
- New approaches for model order reduction required to obtain LPV models suitable for control design.
- Other team members (D. Schmidt, STI, Va. Tech, CMSsoft, Aurora) will play key roles in modeling, design and analysis.
Acknowledgements


Conclusions:

- More efficient, flexible aircraft require integrated flight control systems.
- IQCs can be used in time-domain dissipation-inequalities without loss of conservatism.

Additional Details:

1. http://www.aem.umn.edu/~SeilerControl/
2. http://paaw.net/
Brief Summary of LPV Lower Bound Algorithm

There are many exact results and computational algorithms for LTV and periodic systems (Colaneri, Varga, Cantoni/Sandberg, many others)

The basic idea for computing a lower bound on $\|G_\rho\|$ is to search over periodic parameter trajectories and apply known results for periodic systems.

\[
\|G_\rho\| := \sup_{\rho \in A} \sup_{u \neq 0, u \in L_2} \frac{\|G_\rho u\|}{\|u\|} \geq \sup_{\rho \in A_h} \sup_{u \neq 0, u \in L_2} \frac{\|G_\rho u\|}{\|u\|}
\]

where $A_h \subset A$ denotes the set of admissible periodic trajectories.

Ref: T. Peni and P. Seiler, Computation of lower bounds for the induced $L_2$ norm of LPV systems, submitted to the 2015 CDC.
Simple, 1-parameter LPV system:

\[
\begin{align*}
\delta(t) &\quad \frac{1}{s+1} \\
\delta(t) &\quad \frac{1}{s+1} \\
u(t) &\quad y(t)
\end{align*}
\]

with \(-1 \leq \delta(t) \leq 1\), and \(-\bar{\mu} \leq \dot{\delta}(t) \leq \bar{\mu}\)

The upper bound was computed by searching for a polynomial storage function.
Question: Can this approach be extended to compute lower bounds for uncertain LPV systems?