IQC Analysis of Uncertain LTV Systems
With Rational Dependence on Time

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Abstract—The goal of this paper is to assess the robustness of an uncertain linear time-varying (LTV) system on a finite time horizon. The uncertain system is modeled as a connection of a known LTV system and a perturbation. The state matrices of the LTV system are assumed to be rational functions of time. This is used to model the uncertain LTV system as an connection, denoted by $F_{\Delta}$. Right: Uncertain LTV system, $F_\Delta(H_{LTV}, \Delta_H) = F_u(G_{LT1}, \Delta_G)$.

I. INTRODUCTION

This paper focuses on robustness analysis for a class of uncertain linear-varying (LTV) systems. The analysis is performed on an uncertain LTV system modeled, as shown in Figure 1, by an interconnection of a known, nominal LTV system $H_{LTV}$ and a perturbation $\Delta_H$. This interconnection, denoted by $F_u(H_{LTV}, \Delta_H)$, is a standard tool for uncertainty modeling in robust control [1]. The perturbation can have block structure and is used to model difficult to analyze elements including nonlinearities and dynamic or parametric uncertainty. The analysis performed in this paper characterizes the input-output properties of $\Delta_H$ using integral quadratic constraints (IQC) [2], [3].

This paper presents a finite-dimensional linear matrix inequality (LMI) condition to upper bound the induced $L_2$ gain for this class of uncertain LTV systems. The analysis condition, stated as Theorem 1 in Section III-C, is a variation on the IQC theorem in [2]. There are two key technical issues. First, several IQCs are required to bound time operator $\Delta_t$. These include static IQCs (Section III-A) as well as dynamic IQCs (Section III-B) based on a specific version of the swapping lemma [4]. Second, the main IQC stability theorem in [2] requires the perturbation to be a bounded operator. However, $\Delta_t$ is an unbounded operator on infinite time horizons. Theorem 1 in Section III-C uses a simple dissipation proof that avoids this boundedness assumption. Several simple examples are presented in Section IV to demonstrate the proposed analysis condition.

The most closely related works on robustness of uncertain LTV systems are [5], [6], [7], [8], [9]. These works only assume that the state matrices of the nominal LTV system are piecewise continuous (not necessarily rational) functions of time. This is a more general class of uncertain LTV systems than consider here. The price for this generalization is that the analysis conditions are infinite dimensional (as opposed to the finite dimensional LMIs given here). The work in [8], [9] develops differential, time-dependent LMI analysis conditions. Numerical algorithms are developed by enforcing the LMIs on a finite time grid and using connections to Riccati differential equations. The work in [5] also uses differential LMIs developed for the special case of uncertain LTV systems with a single, full-block uncertainty. Both [6] and [7] propose optimizing over the IQC variables using a Riccati Differential Equation condition.

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The state matrices of the LTV system $H_{LTV}$ are assumed to be rational functions of time. This key assumption yields a useful reformulation for the uncertain system. Specifically, let $\Delta_t$ denote the operator that multiplies a signal by time, i.e. $w_1 = \Delta_t(v_1)$ is defined by $w_1(t) = t \cdot v_1(t)$ for all $t \geq 0$. Then, by assumption, the nominal LTV system $H_{LTV}$ can be modeled by the interconnection of a linear time-invariant (LTI) system $G_{LT1}$ and $\Delta_t$ as shown on the left side of Figure 2. This further implies that the uncertain LTV system in Figure 1 is given by an interconnection of the LTI system $G_{LT1}$ with an augmented perturbation $\Delta_G := [\Delta_t \ 0 \ \Delta_H]$. In other words $F_u(H_{LTV}, \Delta_H) = F_u(G_{LT1}, \Delta_G)$ as shown on the right side of Figure 2.

Fig. 2. Left: Nominal LTV system, $H_{LTV} = F_u(G_{LT1}, \Delta_t)$.

Fig. 1. Interconnection $F_u(H_{LTV}, \Delta_H)$ of a nominal LTV system $H_{LTV}$ and perturbation $\Delta_H$.
II. BACKGROUND

A. Notation

Let $\mathbb{R}^{n \times m}$ and $\mathbb{S}^n$ denote the sets of $n$-by-$m$ real matrices and $n$-by-$n$ real, symmetric matrices. The finite-horizon $L_2$ norm of a signal $v : [0, T] \rightarrow \mathbb{R}^n$ is defined as $\|v\|_2 := (\int_0^T v(t)^T v(t) dt)^{1/2}$. If $\|v\|_2$ is finite then $v \in L_2$. The projection operator $P_T$ maps any function $v$ as follows: $(P_T v)(t) = v(t)$ for $t \leq T$ and $(P_T v)(t) = 0$ otherwise. The extended space, denoted $L_{2e}$, is the set of functions $v$ such that $P_T v \in L_2$ for all $T \geq 0$. $\mathbb{R}L_{\infty}$ denotes the set of rational functions with real coefficients that have no poles on the imaginary axis. $\mathbb{RH}_{\infty}$ is the subset of functions in $\mathbb{R}L_{\infty}$ that are analytic in the closed right-half of the complex plane.

B. Integral Quadratic Constraints (IQCs)

IQCs [2] are used to describe the input/output behavior of an operator $\Delta$. They can be formulated in either the frequency or time domain. The time domain formulation is used in this paper and is based on the graphical interpretation in Figure 3. The inputs and outputs of $\Delta$ are filtered through an LTI system $\Psi$ with zero initial condition $x_\psi(0) = 0$. The dynamics of $\Psi$ are given as follows:

\[
\begin{align*}
\dot{x}_\psi(t) &= A_\psi x_\psi(t) + B_\psi v(t) + B_\varphi w(t), \\
z(t) &= C_\psi x_\psi(t) + D_\psi v(t) + D_\varphi w(t)
\end{align*}
\]

where $x_\psi \in \mathbb{R}^{n_\psi}$ is the state and $(A_\psi, B_\psi, C_\psi, D_\psi)$ denote the state matrices of $\Psi$. Moreover $B_\varphi := [B_{\psi 1}, B_{\psi 2}]$ and $D_\varphi := [D_{\psi 1}, D_{\psi 2}]$ are partitioned conformally with the dimensions of $v$ and $w$. A time domain IQC is an inequality enforced on the output $z$ over finite horizons. The formal definition is given next.

![Fig. 3. Graphical interpretation for time domain IQCs](image)

**Definition 1.** Let $\Psi \in \mathbb{RH}^{n_\Psi \times (n_v + n_w)}_{\infty}$ and $M \in \mathbb{S}^{n_z}$ be given. A bounded, causal operator $\Delta : L_2^{n_v} \rightarrow L_2^{n_w}$ satisfies the time domain IQC defined by $(\Psi, M)$ if the following inequality holds for all $v \in L_2^{n_v}$, $w = \Delta(v)$, and $T \geq 0$:

\[
\int_0^T z(t)^T M z(t) \, dt \geq 0
\]

where $z$ is the output of $\Psi$ driven by inputs $(v, w)$ with zero initial conditions $x_\psi(0) = 0$.

The notation $\Delta \in \mathcal{I}[(\Psi, M)]$ is used when $\Delta$ satisfies the corresponding IQC. Time domain IQCs, as defined above, require the constraint to hold over all finite time horizons. These are often referred to as hard IQCs [2]. An extensive library of IQCs is provided in [2] for various types of perturbations. Most IQCs are specified in the frequency domain using a multiplier II. Under some mild assumptions, a valid time-domain IQC $(\Psi, M)$ can be constructed from II via a $\psi$-spectral factorization [10]. This allows the library of known (frequency domain) IQCs to be used. More general IQC parameterizations are not necessarily “hard” but can be handled with the method in [11].

III. Robustness Analysis

The operator $\Delta_t : L_{2e}^{n_v} \rightarrow L_{2e}^{n_w}$ corresponds to multiplication by time. If $v_1 \in L_{2e}^{n_v}$ and $w_1 = \Delta_t(v_1)$ then $w_1(t) = t \cdot v_1(t)$ for all $t \geq 0$. Note that this operator is passive pointwise in time, i.e. $w_1(t)^T v_1(t) \geq 0$ for all $t \geq 0$. This section exploits additional properties of time multiplication to derive static and dynamic IQCs for $\Delta_t$. Then an LMI condition is presented to bound the induced $L_2$ gain of the uncertain LTV system.

A. Static IQCs for $\Delta_t$

Let $X$ and $Y$ be any $n_v \times n_v$ matrices satisfying $X \succeq 0$ and $Y = -Y^T$. Then $w_2 = \Delta_t(v_1)$ implies:

\[
\begin{pmatrix} v_1(t) \\ w_1(t) \end{pmatrix}^T \begin{pmatrix} 0 & T_1 \\ T_1^T & X + Y^T \end{pmatrix} \begin{pmatrix} v_1(t) \\ w_1(t) \end{pmatrix} = t \cdot v_1(t)^T (2X + Y + Y^T) v_1(t) \geq 0 \quad \forall t \geq 0
\]

Hence $\Delta_t$ satisfies the quadratic constraint in Equation 3 at each point in time. Thus $\Delta_t$ satisfies the time domain IQC defined by $(\Psi, M)$ with

\[
\Psi := I_{2n_v} \quad M := \begin{bmatrix} 0 & X + Y \\ X + Y^T & 0 \end{bmatrix}
\]

This is a static IQC because the filter $\Psi$ contains no dynamics. This IQC holds for all $T \in [0, \infty)$ as required by Definition 1 and hence it can be used for infinite-horizon analysis. It is also useful to define an IQC for finite-horizon analysis. Specifically, let $X$, $Y$, and $Z$ be $n_v \times n_v$ matrices with $X, Z \succeq 0$ and $Y = -Y^T$. Then for any $T_0 \in [0, \infty)$,

\[
\begin{pmatrix} v_1(t) \\ w_1(t) \end{pmatrix}^T \begin{pmatrix} T_0^2 Z & X + Y \\ X + Y^T & -Z \end{pmatrix} \begin{pmatrix} v_1(t) \\ w_1(t) \end{pmatrix} = t \cdot v_1(t)^T (2X + Y + Y^T) v_1(t) + (T_0^2 - t^2) \cdot v_1(t)^T Z v_1(t) \geq 0 \quad \forall t \in [0, T_0]
\]

This implies that $\Delta_t$ satisfies the time domain IQC defined by $(\Psi, M)$ with

\[
\Psi := I_{2n_v} \quad M := \begin{bmatrix} T_0^2 Z & X + Y \\ X + Y^T & -Z \end{bmatrix}
\]

The IQC in Equation 2 only holds for $T \in [0, T_0]$. Hence it technically does not satisfy Definition 1 which requires the IQC to hold for all $T \geq 0$. However, the IQC in Equation 5 can be used to bound the induced $L_2$ gain over the finite horizon $[0, T_0]$.

B. Dynamic IQCs for $\Delta_t$

Dynamic IQCs for $\Delta_t$ are derived in this section as a corollary of the Swapping Lemma [4]. First let $\delta : \mathbb{R} \rightarrow \mathbb{R}$ be...
a differentiable function of time and define the multiplication operator \( \Delta_\delta : L_{2e}^{n+1} \rightarrow L_{2e}^{n+1} \) as follows: if \( v_1 \in L_{2e}^{n+1} \) and \( w_1 = \Delta_\delta(v_1) \) then \( w_1(t) = \delta(t) \cdot v_1(t) \) for all \( t \geq 0 \). The swapping lemma for \( \Delta_\delta \) is stated next.

**Lemma 1.** [4] Let \( F \in \mathbb{R}^{n \times n} \) have a realization with state matrices \((A_F, B_F, C_F, D_F)\). Then

\[
F \Delta_\delta = \Delta_\delta F - F C \Delta_\delta F_B
\]

where the state realizations of \( F_B \) and \( F_C \) are given by \( F_B := (A_F, B_F, I, 0) \) and \( F_C := (A_F, I, C_F, 0) \).

The swapping lemma yields a class of dynamic IQCs for \( \Delta_\delta \) as given in the next corollary.

**Corollary 1.** Let \( F \in \mathbb{R}^{n \times n} \) be given with realization \((A_F, B_F, C_F, D_F)\) and define \( F_B \) and \( F_C \) as in Lemma 1. \( \Delta_\delta \) satisfies the following time-domain IQC for any \( X \geq 0 \) and \( Y = -YT \) of appropriate dimension:

\[
\Psi := \begin{bmatrix} F & 0 \\ F C F_B & F \end{bmatrix} \quad \text{and} \quad M := \begin{bmatrix} 0 & X + Y \\ X + YT & 0 \end{bmatrix}
\]

(6)

**Proof.** The derivative of time is simply one, i.e. \( \Delta_\delta(v_1) = v_1 \). Hence the swapping lemma for \( \Delta_\delta \) simplifies to:

\[
F \Delta_\delta = \Delta_\delta F - F C \Delta_\delta F_B
\]

(7)

Let \( w_1 = \Delta_\delta(v_1) \) for some \( v_1 \in L_{2e}^{n+1} \). Figure 4 provides a graphical interpretation of the swapping lemma result. The input to the dashed box is \( v_1 \) and the output is \( F \Delta_\delta(v_1) = Fw_1 \). This output is equal to \( \Delta_\delta F - F C F_B v_1 \) by Equation 7 and as shown inside the dashed box of Figure 4. The internal signals \( \tilde{v}_1 \) and \( \tilde{w}_1 = \Delta_\delta(\tilde{v}_1) \) satisfy the static quadratic constraint in Equation 3:

\[
\begin{bmatrix} \tilde{v}_1(t)^T \\ \tilde{w}_1(t)^T \end{bmatrix} \begin{bmatrix} 0 & X + Y \\ X + YT & 0 \end{bmatrix} \begin{bmatrix} \tilde{v}_1(t) \\ \tilde{w}_1(t) \end{bmatrix} \geq 0 \quad \forall t \geq 0
\]

(8)

These internal signals satisfy \( \tilde{v}_1 = F \tilde{v}_1 \) and \( \tilde{w}_1 = F C F_B \tilde{v}_1 + Fw_1 \). Substitute these relations into Equation 8 to verify that \( \Delta_\delta \) satisfies the dynamic IQC in Equation 6.

![Fig. 4. Swapping Lemma for \( \Delta_\delta \)](image)

The dynamic IQC in Corollary 1 holds for all \( T \geq 0 \) and can be used for infinite-horizon analysis. A dynamic IQC for analysis on the finite horizon \([0, T_0]\) is given by:

\[
\Psi := \begin{bmatrix} F & 0 \\ F C F_B & F \end{bmatrix} \quad \text{and} \quad M := \begin{bmatrix} T_{0}^{T} Z & X + Y \\ X + YT & -Z \end{bmatrix}
\]

(9)

where \( X, Z \geq 0 \) and \( Y = -YT \). This fact follows similar to Corollary 1 but using the static finite horizon constraint in Equation 4.

### C. Robust Induced \( L_2 \) Gain

The uncertain LTV system is modeled by \( F_u(G_{LT1}, \Delta_G) \) as shown in Figure 2. The nominal system \( G_{LT1} \) is described by the following state-space model:

\[
\begin{align*}
\dot{x}_G(t) &= A_G x_G(t) + B_{G1} w(t) + B_{G2} d(t) \\
v(t) &= C_G x_G(t) + D_{G11} w(t) + D_{G12} d(t) \\
e(t) &= C_G x_G(t) + D_{G21} w(t) + D_{G22} d(t)
\end{align*}
\]

(10)

where \( x_G \in \mathbb{R}^{n_G} \) is the state. The inputs are \( w \in \mathbb{R}^{n_w} \) and \( d \in \mathbb{R}^{n_d} \) while \( v \in \mathbb{R}^{n_v} \) and \( e \in \mathbb{R}^{n_e} \) are outputs. The partitioning \( v := \begin{bmatrix} v_1 \\ w \end{bmatrix} \) and \( w := \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \) is consistent with the structure of the augmented perturbation \( \Delta_G := \begin{bmatrix} \Delta_G & 0 \\ 0 & \Delta_H \end{bmatrix} \).

Well-posedness of the interconnection \( F_u(G_{LT1}, \Delta_G) \) is defined as follows.

**Definition 2.** \( F_u(G_{LT1}, \Delta_G) \) is well-posed if for all \( x_G(0) \in \mathbb{R}^{n_G} \) and \( d \in L_{2e}^{n_d} \) there exists unique solutions \( x_G \in L_{2e}^{n_G} \), \( v \in L_{2e}^{n_v} \), \( e \in L_{2e}^{n_e} \) and \( w \in L_{2e}^{n_w} \) satisfying Equation (10) and \( w = \Delta(v) \) with a causal dependence on \( d \).

The objective is to assess the robustness of the uncertain LTV system \( F_u(G_{LT1}, \Delta_G) \) as shown on the right of Figure 2. For a given \( \Delta_G \), the induced \( L_2 \) gain from \( d \) to \( e \) is defined as:

\[
\| F_u(G_{LT1}, \Delta_G) \| := \sup_{\psi \in L_{2e}^{n_d}, \|d\|_{2} = 1} \|e\|_{2}
\]

(11)

The robustness of \( F_u(G_{LT1}, \Delta_G) \) is analyzed using the interconnection shown in Figure 5. The extended system of \( G_{LT1} \) (Equation 10) and the IQC filter \( \Psi \) (Equation 1) is governed by the following state space model:

\[
\begin{align*}
\dot{x}(t) &= A x(t) + B \begin{bmatrix} w(t) \\ d(t) \end{bmatrix} \\
z(t) &= C_1 x(t) + D_1 \begin{bmatrix} w(t) \\ d(t) \end{bmatrix} \\
e(t) &= C_2 x(t) + D_2 \begin{bmatrix} w(t) \\ d(t) \end{bmatrix}
\end{align*}
\]

(12)

The extended state vector is \( x := \begin{bmatrix} x_G^T \\ x_e^T \end{bmatrix} \in \mathbb{R}^{n} \) where \( n := n_G + n_e \). The state-space matrices are given by:

\[
\begin{align*}
A := \begin{bmatrix} A_G & 0 \\ B_{G1} C_{G1} & A_\psi \end{bmatrix}, & \quad B := \begin{bmatrix} B_{G1} \\ B_{G1} D_{G11} + B_{G2} \end{bmatrix} \\
C_1 := \begin{bmatrix} D_{G11} & C_\psi \end{bmatrix}, & \quad C_2 := \begin{bmatrix} 0 \\ D_{G21} \end{bmatrix} \\
D_1 := \begin{bmatrix} D_{G11} & D_{G12} \end{bmatrix}, & \quad D_2 := \begin{bmatrix} D_{G21} \\ D_{G22} \end{bmatrix}
\end{align*}
\]

The actual system to be analyzed is \( F_u(G_{LT1}, \Delta_G) \) with input \( d \). The analysis is instead performed with the extended LTI system (Equation 12) and the constraint \( \Delta_G \in \mathbb{T}(\Psi, M) \). The constrained extended system has inputs \( (d, w) \). The IQC implicitly constrains the input \( w \) such that the constrained extended system without \( \Delta_G \) includes all behaviors of the original system \( F_u(G_{LT1}, \Delta_G) \).

The following LMI is used to assess the robust perfor-
Proof. The result does not require the perturbation \( \Delta_t \) to be a bounded operator. This is important as \( \Delta_t \) is not a bounded operator on an infinite horizon. Also note that the finite horizon IQCs given in the previous section are valid over the horizon \([0, T_0] \). These can be used to bound the gain of the uncertain LTV system over this finite horizon. The result is similar to Theorem 1 and is not formally stated.

IV. EXAMPLES

A. Nominal LTV Analysis

Consider the following LTV system \( H_{LTV} \)

\[
\dot{x}_H(t) = - \left( 1 + \frac{0.9t}{1+0.9t} \right) x_H(t) + d(t) \quad (17)
\]

\[e(t) = 5x_H(t) \quad (18)\]

This is a nominal LTV system with no uncertainty. The IQCs developed in the previous section can be used to bound the induced \( L_2 \) gain for this LTV system. First, the nominal LTV system is expressed as \( H_{LTV} = F_u(G_{LTI}, \Delta_t) \) where \( G_{LTI} \) is given by:

\[
\dot{x}_G(t) = -x_G(t) - \sqrt{0.9} w_1(t) + d(t) \quad (19)
\]

\[v_1(t) = \sqrt{0.9} x_G(t) - 0.9 w_1(t) \quad (20)\]

\[e(t) = 5x_G(t) \quad (21)\]

The dynamic IQC given in Corollary 1 for \( \Delta_t \) is used with the following filter

\[F(s) := \begin{bmatrix} 1, \frac{1}{s+2}, \ldots, \frac{1}{(s+2)^T} \end{bmatrix}^T \quad (22)\]

The matrices \( X \geq 0 \) and \( Y = -Y^T \) are decision variables solved by optimization. Specifically, given \( \gamma \) is minimized subject to the LMI condition in Theorem 1. This is solved using CVX [16] yielding the bound on the (infinite-horizon) induced \( L_2 \) gain of \( \gamma = 3.18 \).

The dynamic IQC describes a set of uncertainties that include \( \Delta_t \). Hence the LMI condition is only sufficient, i.e. it yields an upper bound on the induced \( L_2 \) gain. There are necessary and sufficient Riccati Differential Equation conditions that can be used to compute exact bounds (within a bisection tolerance) on the induced \( L_2 \) gain of the nominal LTV system [17], [18], [19]. These exact bounds can be used to assess the conservativeness of the dynamic IQC developed for \( \Delta_t \). The Riccati Differential Equation condition yields an induced \( L_2 \) gain of 2.92 on a finite horizon of \([0, 6] \) sec. Longer time horizons yield the same result to two decimal places. The IQC bound of 3.18 is 9% larger than this “exact” result. The conservatism introduced by using the dynamic IQC is relatively small for this example.

To continue this analysis, the induced gain was computed on different finite horizons \([0, T_0] \) with \( T_0 \in \{0.01, 0.1, 0.5, 1, 2, \ldots, 6\} \). The exact results using the Riccati Differential Equation condition are shown as the blue solid curve in Figure 6. A bound on the gain was also computed using the dynamic IQC but with the additional finite horizon term shown in Equation 5. This yields the red dashed curve in Figure 6. It took approximately 22
seconds on a standard laptop to compute the IQC bounds on all nine finite horizons. As noted above the curves for the Riccati and IQC conditions converge to 2.92 and 3.18, respectively as \( T_0 \to \infty \). The IQC bound is slightly larger on all horizons indicating some conservatism in the IQC condition. However, the benefit of the IQC condition is that it can also be used to bound the gain for uncertain LTV systems. This is demonstrated in Section IV-B.

The gain bound computed using the LMI condition in Theorem 1 depends on the chosen filter \( F(s) \). Consider filters of the following form where \( n_F \) is a fixed, positive integer:

\[
F(s) := \left[ \frac{1}{s + 2}, \ldots, \frac{1}{(s + 2)^{n_F - 1}} \right]^T
\]  

(23)

Table I shows the infinite-horizon induced \( L_2 \) gain bounds obtained with the LMI condition in Theorem 1 for several values of \( n_F \). The bound \( \gamma = 3.18 \) given for \( n_F = 8 \) was already reported above. The gain bound steadily improves with increasing values of \( n_F \). The larger (worse) gain bound of \( \gamma = 5.0 \) was found using the static multiplier \( F(s) = 1 \) corresponding to \( n_F = 1 \). We also obtain \( \gamma = 5.0 \) on all finite horizons with the static multiplier \( F(s) = 1 \). It seems, based on this one example, that static multipliers are ineffective for finite-horizon analysis. No formal proof of this statement has been obtained as of yet. Additional details on IQC parameterizations can be found in [3].

\[
\begin{array}{c|cccccc}
 n_F & 1 & 2 & 3 & 4 & 8 \\
 \hline
 \gamma & 5.00 & 3.71 & 3.37 & 3.25 & 3.18 \\
\end{array}
\]

Table I

INFINITE HORIZON INDUCED \( L_2 \) GAIN BOUNDS USING \( F(s) \) IN Eq. 23

B. Robust LTV Analysis

Consider the feedback system shown in Figure 7. The plant \( P \) is given by the following second-order, LTV system:

\[
\begin{align*}
\dot{x}_P(t) &= \begin{bmatrix} 0 & 1 \\ -4 & -2\zeta(t) \end{bmatrix} x_P(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} 4 & 0 \end{bmatrix} x_P(t)
\end{align*}
\]

(24)

(25)

where \( \zeta(t) := 0.2 + \frac{0.634}{1+0.9t} \). The controller is a simple proportional gain \( K = 2 \). The uncertainty \( \Delta_H \) is assumed to be a causal operator with norm-bound \( ||\Delta_H|| \leq b \). The goal is to compute a bound on the gain from disturbance \( d \) to error \( e \). The closed-loop uncertain LTV system has a rational dependence on time and hence it can be modeled by \( F_u(G_{LT}, \Delta_G) \) as shown on the right of Figure 2.

![Fig. 7. Uncertain LTV Feedback System](image)

The dynamic IQC for \( \Delta_1 \) (Corollary 1) is used with the filter \( F(s) \) given in Equation 22. The following static IQC is used for \( \Delta_H \):

\[
\Psi := I_2 \text{ and } M := \begin{bmatrix} 0 & 1 \\ 0 & \frac{1}{2} \end{bmatrix}
\]

(26)

An (infinite-horizon) upper bound on the induced \( L_2 \) gain was computed using the LMI condition in Theorem 1. Figure 8 shows the gain bound for twelve different values of the uncertainty bound \( b \). It took approximately 24 seconds on a standard laptop to compute these IQC bounds. The IQC condition yields a bound on the nominal gain (\( b = 0 \)) of 2.09. For comparison, the Riccati Differential Equation condition yields an induced \( L_2 \) gain of 1.90 on a finite horizon of [0, 10] sec. Longer time horizons yield the same value to two decimal places. This value is shown as the red circle in Figure 8. The IQC condition yields a nominal gain that is about 10% higher than the exact value from the Riccati Differential Equation. However, the IQC condition can also be used to assess the robust performance when the uncertainty \( \Delta_H \) is introduced into the feedback loop.

V. CONCLUSIONS

This paper derived LMI conditions to bound the induced \( L_2 \) gain of an uncertain LTV system. The uncertain system was assumed to have state matrices with rational dependence on time. This enables the uncertain system to be formulated as an interconnection of a time invariant system and an augmented uncertainty that includes time. Static and dynamic integral quadratic constraints were introduced to bound the time operator. These can be used as part of a dissipation inequality condition to bound the induced \( L_2 \) gain on either finite or infinite horizons. Two simple examples were provided to demonstrate the approach. Future work will include additional studies on the conservatism of...
the proposed approach and possible extensions to synthesis conditions.

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