Robust Synthesis for Linear Parameter Varying Systems Using Integral Quadratic Constraints

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Abstract

A robust synthesis algorithm is developed for a class of uncertain, linear parameter varying (LPV) systems. The uncertain system is described as an interconnection of a nominal LPV system and a block structured uncertainty. The nominal part is a “gridded” LPV system with state matrices that are arbitrary functions of the parameter. The input/output behavior of the uncertainty is described by integral quadratic constraints (IQCs). The robust synthesis problem leads to a non-convex optimization. The proposed algorithm is a coordinate-wise descent similar to the well-known DK iteration for $\mu$ synthesis. It alternates between an LPV synthesis step and an IQC analysis step. Both steps can be efficiently solved as semidefinite programs. The derivation of the synthesis algorithm is less obvious for LPV systems as compared to its LTI counterpart due to the lack of a valid frequency response interpretation. The main contribution is the construction of the iterative synthesis algorithm using time domain dissipation inequalities and a scaled system analogous to that appearing in $\mu$ synthesis. It is shown that the proposed algorithm ensures that the robust performance is non-increasing at each iteration step. The effectiveness of the proposed method is demonstrated on a simple numerical example.

1 INTRODUCTION

This paper considers the robust synthesis problem for a class of uncertain linear parameter varying (LPV) systems. The uncertain system is described as an interconnection of a nominal (not-uncertain) LPV system and a block structured uncertainty. The state matrices of the nominal system have an arbitrary dependence on parameters, i.e. it is a “gridded” LPV system. Such models arise naturally in many applications via linearization of a nonlinear model around parameterized operating (trim) points (Moreno et al., 2012; Bobanac et al., 2010). The input/output behavior of the uncertainty is described by integral quadratic constraints (IQCs) (Megretski and Rantzer, 1997). The use of IQCs is sufficiently general to describe “uncertain” components that include nonlinearities, in addition to (parametric or dynamic) uncertainty.

The robust synthesis problem, formulated in Section 3.1, is to synthesize a controller that minimizes a closed-loop robust performance metric. This leads to a non-convex optimization that involves a search for both the controller state matrices and the IQC analysis variables. The proposed algorithm, given in Section 3.2, consists of a coordinate-wise descent similar to the well-known DK-iteration (Zhou et al., 1996; Balas et al., 2007) for $\mu$ synthesis. Specifically, the proposed algorithm alternates between an LPV synthesis step and an IQC analysis step. The synthesis step essentially relies on existing results for nominal LPV systems in Wu et al. (1996). The analysis step is performed using a matrix inequality condition to bound the robust performance of the closed-loop uncertain LPV system (Section 4.1). Both steps can be efficiently solved as semidefinite programs (SDPs). The effectiveness of the proposed method is demonstrated on a numerical example in Section 5.

There are two main technical challenges. First, the nominal LPV system does not have a valid frequency response interpretation and hence the analysis requires a time domain approach. Section 4.1 develops a matrix inequality robustness analysis condition (Theorem 2) using (time domain) dissipation inequality techniques. This analysis condition is an extension of the worst-case gain condition in Pfifer and Seiler (2014). An alternative to the dissipation inequality based approach for IQCs in the time domain is given in Cantoni et al. (2013). It is purely based on operator theory and uses homotopy arguments to prove stability. This alternative approach can potentially be used to develop synthesis algorithms complementary to the one developed here or provide an alternative proof for the presented algorithm. The second
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2.1 Linear Parameter Varying (LPV) Systems

LPV systems are a class of systems whose state-space

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parameter \( \rho : \mathbb{R}^+ \rightarrow \mathbb{R}^{n_\rho} \). The set of admissible parameter

trajectories is defined as \( \mathcal{T} := \{ \rho : \mathbb{R}_+ \rightarrow \mathbb{R}^{n_\rho} : \rho(t) \in \mathcal{P} \forall t \geq 0 \text{ and } \rho(t) \text{ is continuously differentiable} \} \) where \( \mathcal{P} \subset \mathbb{R}^{n_\rho} \) is a known compact set. In some applications, the parameter varying rate \( \dot{\rho} \) are assumed to be bound-

ed. However, only the rate unbounded case is considered here for simplicity. All results in this paper generalize, but with extensive notations, to the rate bounded case using existing results in (Wu et al., 1996; Pfifer and Seiler, 2014). An \( n^{\rho} \) order LPV system, \( G_{\rho} \), is defined by

\[
\dot{x} = \begin{bmatrix} A(\rho) & B(\rho) \\ C(\rho) & D(\rho) \end{bmatrix} x + \begin{bmatrix} 0 \\ u \end{bmatrix}
\]

where \( A : \mathcal{P} \rightarrow \mathbb{R}^{n_x \times n_x}, B : \mathcal{P} \rightarrow \mathbb{R}^{n_x \times n_u}, C : \mathcal{P} \rightarrow \mathbb{R}^{n_y \times n_x} \), and \( D : \mathcal{P} \rightarrow \mathbb{R}^{n_y \times n_u} \). The performance of an LPV system \( G_{\rho} \) is specified by its induced \( L_2 \) gain

\[
\|G_{\rho}\| := \sup_{\rho \neq \rho_0 \in \mathcal{P}_e} \left\| \begin{bmatrix} x \\ d \end{bmatrix} \right\|_{\mathcal{L}_2}.
\]

A generalization of the Bounded Real Lemma Wu et al. (1996) provides a suffi-
cient condition to upper bound \( \|G_{\rho}\| \). The next theorem states the condition but simplified for the special case of rate unbounded LPV systems.

Theorem 1. (Wu et al. (1996)): \( G_{\rho} \) is exponentially stable and \( \|G_{\rho}\| \leq \gamma \) if there exists \( P = P^T \succeq 0 \) such that \( \forall \rho \in \mathcal{P} \)

\[
\begin{bmatrix} PA(\rho) + A(\rho)^T P & PB(\rho) \\ B^T(\rho) P & -I \end{bmatrix} + \frac{1}{\gamma^2} \begin{bmatrix} C(\rho) & D(\rho) \end{bmatrix} \begin{bmatrix} C(\rho) & D(\rho) \end{bmatrix} \succeq 0
\]

This theorem forms the basis for the induced \( L_2 \) norm controller synthesis in Wu et al. (1996). Consider an open loop LPV system \( G_{\rho} \) with inputs \( [d^T, u^T]^T \) and outputs \( [e^T, y^T]^T \). The objective is to synthesize a controller \( K_{\rho} \):

\[
\begin{bmatrix} \dot{x}_K \\ y \end{bmatrix} = \begin{bmatrix} A_K(\rho) & B_K(\rho) \\ C_K(\rho) & D_K(\rho) \end{bmatrix} \begin{bmatrix} x_K \\ u \end{bmatrix}
\]

such that the closed-loop interconnection of \( G_{\rho} \) and \( K_{\rho} \), which is given by the lower linear fractional transfor-
mation (LFT), denoted \( \mathcal{F}_1(G_{\rho}, K_{\rho}) \), has the minimal induced \( L_2 \) gain: \( \min_{\rho \in \mathcal{P}_e} \|G_{\rho}K_{\rho}\|_2 \). This LPV synthesis problem can be solved via parameterized LMI conditions.

2.2 Integral Quadratic Constraints (IQCs)

IQCs (Megretski and Rantzer, 1997) provide a framework for robustness analysis building on work by Yakubovich (1971). The IQC specifies a constraint on the input/output signals of the perturbation. The following definitions characterize the constraint in the frequency and time domain.

Definition 1. Let \( \Pi \in \mathbb{R}_+^{L_2^{n_u} \times L_2^{n_u}} \) be given. Two signals \( v \in L_2^{n_u} [0, \infty) \) and \( w \in L_2^{n_u} [0, \infty) \) satisfy the frequency domain IQC defined by the multiplier \( \Pi \) if

\[
\int_{-\infty}^{\infty} \left[ \hat{V}(j\omega) \hat{W}(j\omega) \right] \Pi(j\omega) \left[ \hat{V}(j\omega) \hat{W}(j\omega) \right]^* d\omega \geq 0
\]

where \( \hat{V} \) and \( \hat{W} \) are Fourier transforms of \( v \) and \( w \). A bounded, causal operator \( \Delta : L_2^{n_u} [0, \infty) \rightarrow L_2^{n_u} [0, \infty) \) satisfies the frequency domain IQC defined by \( \Pi \) if Eq. 4 holds for all \( v \in L_2^{n_u} [0, \infty) \) and \( \Delta = (v, w) \).

Definition 2. Let \( \Psi \) be a stable LTI system, i.e. \( \Psi \in \mathbb{R}_+^{L_2^{n_y} \times L_2^{n_u}} \), and \( M = M^T \in \mathbb{R}_+^{n_y \times n_u} \). Two signals \( v \in L_2^{n_u} [0, \infty) \) and \( w \in L_2^{n_u} [0, \infty) \) satisfy the time domain IQC defined by the multiplier \( \Psi \) and matrix \( M \) of the following inequality holds for all \( T \geq 0 \)

\[
\int_{0}^{T} z^T(t)Mz(t)dt \geq 0
\]

where \( z \) is the output of \( \Psi \) driven by inputs \((v, w)\) with zero initial conditions. A bounded, causal operator \( \Delta : L_2^{n_u} [0, \infty) \rightarrow L_2^{n_y} [0, \infty) \) satisfies the time domain IQC
defined by \((\Psi, M)\) if Eq. 5 holds for all \(v \in L^2_{w_0}[0, \infty),\)
\(w = \Delta(v)\) and \(T \geq 0\).

IQC\s can be used to model a variety of nonlinearities and uncertainties, e.g. saturation and norm bounded uncertainty (Megretski and Rantzer, 1997). Fig. 1 provides a graphical interpretation for the time domain IQC. If \(\Delta\) satisfies the time domain IQC defined by \(\Psi\) then the filtered signal \(z\) satisfies the constraint in Eq. 5 for any finite-horizon \(T \geq 0\).

![Fig. 1. Graphical interpretation of the IQC](image)

A precise connection between the frequency and time domain IQC formulations is important for the robust synthesis algorithm described in this paper. Specifically, if \(\Delta\) satisfies the time domain IQC defined by \((\Psi, M)\) then it satisfies the frequency domain IQC defined by \(\Pi = \Psi^* M \Psi\). However, the reverse implication fails to hold in general (Seiler, 2015). A time domain IQC as in Definition 2 is referred to as a hard IQC in Megretski and Rantzer (1997). In contrast, factorizations for which the time domain constraint holds only for \(T = \infty\) are called soft IQCs. Lemma 4 in Seiler (2015) provide a specific “hard” factorization \((\Psi, M)\) (called a J-spectral factorization of \(\Pi\)) that can be constructed under additional assumptions on the frequency domain multiplier \(\Pi\). The distinction between hard and soft IQCs is important because LPV systems do not have a valid frequency response interpretation. Hence existing conditions for robust analysis of gridded LPV systems (Pfifer and Seiler, 2014) rely on the use of valid time domain (hard) IQCs. Section 4.1 generalizes these existing results to handle factorizations \((\Psi, M)\) that are not necessarily hard.

3 Robust Synthesis

3.1 Problem Formulation

The robust synthesis problem involves an uncertain system (Fig. 2) described by the interconnection of an LPV system \(G_p\), a perturbation \(\Delta\), and an LPV controller \(K_p\). The state-space realization for \(G_p\) is given by:

\[
\begin{bmatrix}
\dot{x}_G \\
\dot{v} \\
\dot{e} \\
\dot{y}
\end{bmatrix} =
\begin{bmatrix}
A(p) & B_w(p) & B_d(p) & B_u(p) \\
C_v(p) & D_{vw}(p) & D_{vd}(p) & D_{vu}(p) \\
C_v(p) & D_{ew}(p) & D_{ed}(p) & D_{eu}(p) \\
C_y(p) & D_{yw}(p) & D_{yd}(p) & D_{yu}(p)
\end{bmatrix}
\begin{bmatrix}
x_G \\
v \\
e \\
y
\end{bmatrix}
\]

where \(x_G \in \mathbb{R}^{n_x}, v \in \mathbb{R}^{n_w}, d \in \mathbb{R}^{n_d}, u \in \mathbb{R}^{n_u}, v \in \mathbb{R}^{n_v}, e \in \mathbb{R}^{n_e}\) and \(y \in \mathbb{R}^{n_y}\). The following assumptions are made regarding \(G_p\) and \(\Delta\):

**Assumption 1.** \(G_p\) is quadratically stabilizable from \(u\) and quadratically detectable from \(y\) as defined in Chapter 1 of Wu (1995).

**Assumption 2.** The perturbation is a bounded, causal operator \(\Delta : L^2_{w_0}[0, \infty) \to L^2_{v_0}[0, \infty]\) that satisfies a collection of frequency domain IQCs defined by \(\Pi_k^{N} = \Pi^{N}_{k=1} \subset \mathbb{R}[\omega]^{(n_u+n_w) \times (n_e+n_u)}\), denoted \(\Delta \in \Delta(\Pi_1, \ldots, \Pi_N)\).

**Assumption 3.** Partition the frequency domain multipliers \(\Pi_k^{N} = \Pi_{k,11}^{N}, \Pi_{k,22}^{N}\) where \(\Pi_{k,11}\) is a hard IQC and has a “hard” factorization \(\Pi_{k,11} \). Each frequency domain multiplier satisfies \(\Pi_{k,11}(j \omega) \geq 0\) and \(\Pi_{k,22}(j \omega) \leq 0\), for every \(\omega \in \mathbb{R}\). Assumption 3 and 4 are used to simplify the algorithm. Assumption 3 only requires the non-strict definiteness conditions \(\Pi_{k,11} \geq 0\) and \(\Pi_{k,22} \leq 0\). This is sufficiently general to cover most typical frequency domain multipliers used in IQC analysis (Megretski and Rantzer, 1997). However, Assumptions 3 and 4 are used to ensure that a “combined” multiplier \(\Pi := \sum_{k=1}^{N} \lambda_k \Pi_k\) that appears in the proposed robust synthesis algorithm satisfies the strict definiteness conditions by forcing \(\lambda_1 > 0\) and \(\lambda_k \geq 0\) for \(k = 2, \ldots, N\). Therefore, \(\Pi\) is a hard IQC and has a J-spectral factorization (Theorem 4 in Seiler (2015)).

To simplify notation, define \(H_p := \mathcal{F}_I(G_p, K_p)\). The uncertain LPV system in Fig. 2 can therefore be expressed as an upper LFT, denoted \(\mathcal{F}_u(H_p, \Delta)\). A natural performance metric for the uncertain LPV system is the worst-case gain:

\[
sup_{\Delta \in \Delta(\Pi_1, \ldots, \Pi_N)} \|\mathcal{F}_u(H_p, \Delta)\|.
\]

This is the largest induced \(L_2\) gain of the uncertain LPV system over all uncertainties consistent with the specified IQCs. This metric has been widely used for robustness analysis (Pfifer and Seiler, 2014; Turner and Kerr, 2012). However, it is inconvenient for robust synthesis as it requires an initial controller that achieves robust stability with respect to \(\Delta(\Pi_1, \ldots, \Pi_N)\). Thus it is standard, e.g. in D-K synthesis, to instead use a robust performance metric that simultaneously scales both the uncertainty level and the system gain. The definition of robust performance
requires the notion of a scaled uncertainty set. Specifically, define $S_k$ as the scaling matrix $\text{diag}(b_{1k}, I_{n_k})$. Let $\Delta_k(\Pi_1, \ldots, \Pi_N)$ denote the set of bounded, causal operators $\Delta$ that satisfy the frequency domain IQCs defined by $S_k \Pi_k S_k$ for $k = 1, \ldots, N$. For the scaled set, if $b_2 \geq b_1$ then $\Delta_{b_2} \supseteq \Delta_{b_1}$. The definition of robust performance uses this scaled uncertainty set.

**Definition 3.** The system $H_\rho$ achieves robust performance of level $\gamma$ with respect to the uncertainty described by $\{\Pi_k\}_{k=1}^N$ if

$$\sup_{\Delta \in \Delta_{1/\gamma}(\Pi_1, \ldots, \Pi_N)} \| F_\rho(H_\rho, \Delta) \| \leq \gamma \quad (7)$$

Let $r_{\Delta_{1/\gamma}(\Pi_1, \ldots, \Pi_N)}[H_\rho]$ denote the smallest level of robust performance achievable by $H_\rho$. $H_\rho$ achieves robust performance of level $\gamma$ if the worst-case gain is $\leq \gamma$ over all uncertainties in the scaled set $\Delta_{1/\gamma}(\Pi_1, \ldots, \Pi_N)$. For decreasing levels of robust performance, the gain decreases and the bound on the tolerable uncertainty increases. The robust synthesis problem is to synthesize a controller $K_\rho$ that stabilizes $G_\rho$ and minimizes the closed-loop robust performance, i.e.:

$$\inf_{K_\rho, \text{stabilizing}} r_{\Delta(\Pi_1, \ldots, \Pi_N)}[F_\rho(G_\rho, K_\rho)] \quad (8)$$

### 3.2 Algorithm

This section gives a high-level overview of the proposed LPV robust synthesis algorithm. Detailed steps are described in Algorithm 1. Technical results regarding the algorithm are given in Section 4. As in DK synthesis, the robust LPV synthesis is non-convex. In particular, Theorem 2 in Section 4.1 provides a linear matrix inequality (LMI) formulation for robust performance. Applying this result for synthesis leads to a matrix inequality that is bilinear in the state matrices for $K_\rho$ and the analysis variables consisting of a storage matrix $P \succeq 0$ and IQC coefficients $\{\lambda_k\}_{k=1}^N$. A coordinate-wise approach is used to decouple the design into a nominal controller synthesis step (for $K_\rho$) and a robust performance analysis step (for $P$ and $\lambda$). The technical results in Section 4 are used to link these steps. The proposed algorithm will not, in general, converge to a local (nor global) optima. However, it is a pragmatic approach that decouples the synthesis and analysis steps into convex optimizations. The main technical result (Theorem 3 in Section 4.3) is that the algorithm iteration is well posed at each step and the robust performance is non-increasing. This property is similar to DK synthesis.

### 4 Technical Results

#### 4.1 Robust Performance Condition

This section derives a matrix inequality condition to bound the robust performance for an uncertain LPV system $F_\rho(H_\rho, \Delta)$. The nominal LPV system $H_\rho$ has the following state-space realization:

![Fig. 3. LFT Interconnection of Scaled System, $G_\rho^{\text{sccl}}$](image)

**Algorithm 1 Robust Synthesis for LPV Systems**

1: **Given:** LPV system $G_\rho$ and multipliers $\{\Pi_k\}_{k=1}^N$ satisfying Assumptions 1-4; Stopping tolerance parameters $i_{\text{max}} \in \mathbb{N}$ and $\epsilon_{\text{tol}} > 0$.
2: **Initialization:** Initialize the iteration count to $i := 0$. Set $\lambda(0) = [1, 0, \ldots, 0] \in \mathbb{R}^N_{>0}$ and $\gamma(0) = +\infty$. Factorize each $\Pi_k$ as $(\Psi_k, M_k)$ with $\Psi_k \in \mathbb{R}^{n_k \times (n_{\nu}+n_\omega)}$ according to Section 7.3 of Francis (1987).
3: if $i < i_{\text{max}}$ then
4: **Iteration Count:** Increment count $i := i + 1$.
5: **Performance Scaling:** If $i > 1$ then define the scaling matrix $S(i-1) := \text{diag}(\gamma(1-i)I_{n_\nu}, I_{n_\omega})$, otherwise $S(0) := I_{n_\nu+n_\omega}$.
6: **Combined Multiplier:** Construct $\Pi_\Lambda := \sum_{k=1}^N \lambda_k(i-1)S(i-1)\Pi_kS(i-1)$. Compute a $J$-spectral factorization $(\Psi_\Lambda, M_\Lambda)$ of $\Pi_\Lambda$ according to Lemma 4 in Seiler (2015).
7: **Scaled System Construction:** Invert the $w/\omega$ channels of $\Psi_\Lambda$ (Eq. 22) to construct $\Psi_\Lambda^\dagger$ with (Eq. 23). Form the (open-loop) scaled system $G_\rho^{\text{sccl}}$ by interconnecting $G_\rho$ and $\Psi_\Lambda^\dagger$ as shown in Fig. 3.
8: **Synthesis Step:** Use results in Wu (1995); Wu et al. (1996) to solve the synthesis problem:

$$\min_{K_\rho} \left\{ \| F_\rho(G_\rho^{\text{sccl}}, K_\rho) \| : \text{stabilizing} \right\}$$

This minimizes the (upper bound) on the closed-loop induced gain from $(w_\Lambda, d)$ to $(\omega_\Lambda, e)$. The result is a bound on controller $K_\rho(i)$.
9: **Analysis Step:** Use Theorem 2 in Section 4.1 to compute the best upper bound on the robust performance of the closed-loop $H_\rho := F_\rho(G_\rho, K_\rho(i))$ with respect to $\Delta(\Pi_1, \ldots, \Pi_N)$. The result is the robust performance bound $\gamma(i)$, scalars $\{\lambda_k(i)\}_{k=1}^N$, and storage function matrix $P(i) = P(i)^T$.
10: **Termination Condition:** If $\gamma(i) - \gamma(i-1) \leq \epsilon_{\text{tol}}$ then stop the iteration.
11: **end if**
12: **Return:** Final controller $K_\rho(i)$ and robust performance upper bound $\gamma(i)$. 


where \( x_H \in \mathbb{R}^{n_H} \), \( w \in \mathbb{R}^{n_w} \), \( d \in \mathbb{R}^{n_d} \), \( v \in \mathbb{R}^{n_v} \), and \( e \in \mathbb{R}^{n_e} \). The uncertainty \( \Delta \) is assumed to satisfy Assumptions 2-4 in Section 3.1. Construct a factorization for each \( \Pi_k \) as \((\Psi_k, M_k)\) where \( \Psi_k \) is stable, e.g., using the basic method described in Section 7.3 of Francis (1987). It is emphasized that the factorization \((\Psi_k, M_k)\) need not specify a valid time domain IQC as given by Definition 2. Robust performance is defined with the scaled uncertainty set \( \Delta_{1/\gamma} := \Pi_1 \Pi_2 \cdots \Pi_N \) corresponding to scaled multipliers \( S_{1/\gamma} \Pi_{1/\gamma} \). A factorization for each scaled multiplier is given by \((\Psi_k S_{1/\gamma}, M_k)\). Let \( z_k \) denote the output of the scaled system \( \Psi_k S_{1/\gamma} \), driven by the input/output signals \((v, w)\). Then \( \{\Psi_k S_{1/\gamma}\}_{k=1}^N \) can be aggregated into a single system denoted \( \Psi_{1/\gamma} \), with the following (minimal) state-space realization:

\[
\begin{bmatrix}
\dot{x}_H(t) \\
\dot{e}(t)
\end{bmatrix} =
\begin{bmatrix}
A(\rho) & B_w(\rho) & B_d(\rho) \\
C_v(\rho) & D_{cw}(\rho) & D_{cd}(\rho) \\
C_e(\rho) & D_{ew}(\rho) & D_{ed}(\rho)
\end{bmatrix}
\begin{bmatrix}
x_H(t) \\
e(t)
\end{bmatrix} +
\begin{bmatrix}
w(t) \\
d(t)
\end{bmatrix} \quad (9)
\]

Eq. 10 uses an abbreviated notation to denote the outputs of \( \Psi_{1/\gamma} \) as \([z_{1/\gamma}]^T \). The robust performance analysis is based on Fig. 4 with \( \Delta = \Delta_{1/\gamma} := \Pi_1 \Pi_2 \cdots \Pi_N \). This interconnection is described by \( w = \Delta(v) \) and the extended system of \( H_p \) and \( \Psi_{1/\gamma} \):

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{z}_k(t)
\end{bmatrix} =
\begin{bmatrix}
A(\rho) & B_w(\rho) & B_d(\rho) \\
C_z(\rho) & D_{zw}(\rho) & D_{zd}(\rho) \\
C_e(\rho) & D_{ew}(\rho) & D_{ed}(\rho)
\end{bmatrix}
\begin{bmatrix}
x(t) \\
z_k(t) \\
e(t)
\end{bmatrix} +
\begin{bmatrix}
w(t) \\
d(t)
\end{bmatrix} \quad (10)
\]

where the state vector is \( x = [x_H; x_v] \in \mathbb{R}^{n_H+n_v} \). The extended system can be expressed in terms of the state matrices for \( H_p \) (Eq. 9) and \( \Psi_{1/\gamma} \) (Eq. 10) as:

\[
\Pi_\lambda = \begin{bmatrix}
\ldots \\
\Pi_k \end{bmatrix} \begin{bmatrix}
A(\rho) & B_w(\rho) & B_d(\rho) \\
C_v(\rho) & D_{cw}(\rho) & D_{cd}(\rho) \\
C_e(\rho) & D_{ew}(\rho) & D_{ed}(\rho)
\end{bmatrix}
\begin{bmatrix}
x_H(t) \\
e(t)
\end{bmatrix} \quad (12)
\]

where \( \bar{B} := \gamma^{-1} \bar{B}_w \bar{B}_w \) and

\[
\begin{bmatrix}
\bar{Q}_\lambda \\
\bar{S}_\lambda
\end{bmatrix} := \sum_{k=1}^N \lambda_k \begin{bmatrix}
\bar{C}_z^T \\
\bar{D}_{zw}
\end{bmatrix} M_k \begin{bmatrix}
\bar{C}_{z_k} \gamma^{-1} \bar{D}_{z_kw} \bar{D}_{z_kw}
\end{bmatrix}
\]

The conditions on \( \{\lambda_k\}_{k=1}^N \) along with Assumptions 3 and 4 imply \( (\Pi_\lambda)_{11}(j\omega) > 0 \) and \( (\Pi_\lambda)_{22}(j\omega) < 0 \) \( \forall \omega \in \mathbb{R} \cup \{+\infty\} \). Thus \( \Pi_\lambda \) has a J-spectral factorization (Lemma 4 in Seiler (2015)). This factorization is constructed from the stabilizing solution X to the ARE in Eq. A.2 with \( (A, B, Q_\lambda, R_\lambda) \). Let \( (\Psi, M) \) be a J-spectral factorization of \( \Pi_\lambda \) with \( M := \text{diag}(I, -I) \). Then \( (\Psi, M) \) can be defined as \((\Psi, S_{1/\gamma}, M_{1/\gamma})\) is another factorization of \( \Pi_\lambda \). This rescaled factorization has \( M_{1/\gamma} := \text{diag}(\gamma^{-2} I, -I) \). A state-space realization for the rescaled filter \( \Psi_{1/\gamma} \) is:

\[
\begin{bmatrix}
\dot{x}_H(t) \\
\dot{z}_k(t)
\end{bmatrix} =
\begin{bmatrix}
A(\rho) & B_w(\rho) & B_d(\rho) \\
C_z(\rho) & D_{zw}(\rho) & D_{zd}(\rho) \\
C_e(\rho) & D_{ew}(\rho) & D_{ed}(\rho)
\end{bmatrix}
\begin{bmatrix}
x_H(t) \\
z_k(t) \\
e(t)
\end{bmatrix} +
\begin{bmatrix}
w(t) \\
d(t)
\end{bmatrix} \quad (14)
\]

where \( \Psi_{1/\gamma} \) is a valid time domain IQC for \( \Delta \) (Seiler, 2015). Finally, an extended system of \( H_p \) and \( \Psi_{1/\gamma} \) can be formed:

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{z}_k(t)
\end{bmatrix} =
\begin{bmatrix}
A(\rho) & B_w(\rho) & B_d(\rho) \\
C_z(\rho) & D_{zw}(\rho) & D_{zd}(\rho) \\
C_e(\rho) & D_{ew}(\rho) & D_{ed}(\rho)
\end{bmatrix}
\begin{bmatrix}
x(t) \\
z_k(t) \\
e(t)
\end{bmatrix} +
\begin{bmatrix}
w(t) \\
d(t)
\end{bmatrix} \quad (15)
\]

This extended system can be expressed in terms of the state matrices for \( H_p \) (Eq. 9) and \( \Psi_{1/\gamma} \) (Eq. 14).

Two extended systems have been presented thus far. The extended system of \( H_p \) and \( \Psi_{1/\gamma} \) (Eq. 11) can be used to define the following parameterized matrix inequality involving multiple IQCs (neglecting dependence on \( \rho \)):

\[
\begin{bmatrix}
P_A + A^T P \bar{P}_E \bar{P}_B \bar{P}_D \\
P_E^T P \bar{P}_A \bar{P}_B \bar{P}_D \\
P_D^T P \bar{P}_A \bar{P}_E \bar{P}_B \bar{P}_D
\end{bmatrix}
\begin{bmatrix}
C_z^T \\
D_{zw} \\
D_{z_kw}
\end{bmatrix}
\begin{bmatrix}
P_A + A^T P \bar{P}_E \bar{P}_B \bar{P}_D \\
P_E^T P \bar{P}_A \bar{P}_B \bar{P}_D \\
P_D^T P \bar{P}_A \bar{P}_E \bar{P}_B \bar{P}_D
\end{bmatrix}
= 0
\]

\[
+ \sum_{k=1}^N \lambda_k \begin{bmatrix}
\bar{C}_z^T \\
\bar{D}_{zw}
\end{bmatrix} M_k \begin{bmatrix}
\bar{C}_{z_k} \gamma^{-1} \bar{D}_{z_kw} \bar{D}_{z_kw}
\end{bmatrix} < 0
\]

Similarly, the extended system of \( H_p \) and \( \Psi_{1/\gamma} \) (Eq. 15) defines an inequality with the single, rescaled IQC:

\[
\begin{bmatrix}
P_A + A^T P \bar{P}_E \bar{P}_B \bar{P}_D \\
P_E^T P \bar{P}_A \bar{P}_B \bar{P}_D \\
P_D^T P \bar{P}_A \bar{P}_E \bar{P}_B \bar{P}_D
\end{bmatrix}
\begin{bmatrix}
C_z^T \\
D_{zw} \\
D_{z_kw}
\end{bmatrix}
\begin{bmatrix}
P_A + A^T P \bar{P}_E \bar{P}_B \bar{P}_D \\
P_E^T P \bar{P}_A \bar{P}_B \bar{P}_D \\
P_D^T P \bar{P}_A \bar{P}_E \bar{P}_B \bar{P}_D
\end{bmatrix}
= 0
\]

\[
+ \sum_{k=1}^N \lambda_k \begin{bmatrix}
\bar{C}_z^T \\
\bar{D}_{zw}
\end{bmatrix} M_k \begin{bmatrix}
\bar{C}_{z_k} \gamma^{-1} \bar{D}_{z_kw} \bar{D}_{z_kw}
\end{bmatrix} < 0
\]

The technical result regarding these two matrix inequalities is formally stated in the next lemma.
Lemma 1. Let \( \{ \Pi_k \}_{k=1}^N \subset \mathbb{R}^{(n_u+n_v) \times (n_u+n_v)} \) \( q \geq 0 \), and \( \{ \lambda_k \}_{k=1}^N \) be given where \( \{ \Pi_k \}_{k=1}^N \) satisfies Assumptions 3 and 4, \( \lambda_k \in \mathbb{R}_{>0} \) and \( \lambda_k \in \mathbb{R}_{>0} \) \( k \in \{ 2, \ldots, N \} \).

Let each \( \Pi_k \) have a factorization \( (\Psi_k, M_k) \) where \( \Psi_k \) is the stabilizing ARE solution used to construct this factorization. As described above, \( \Pi \) and \( \Psi \) are positive definite and thus does not necessarily define a valid storage function. Lemma 1 addresses both issues. It converts the original problem to an alternative form (Eq. 17) involving only a single, valid time domain IQC.

\[ V(t) = x^T(t)M_xz(t) \leq d(t)^T d(t) - \gamma^{-2} e(t)^T e(t) \]

Append \( \Psi \) to the \( (v, w) \) channels of the uncertain system \( F_u(H_\rho, \Delta) \). This corresponds to the interconnection shown in Fig. 4 except with \( \Psi \) replacing \( \Psi_{1/\gamma} \). Let \( (w, d, z, e) \) be the solution of this interconnection for some \( \Delta \in \Delta_{1/\gamma} \), disturbance \( d \in L^2_\gamma \), admissible trajectory \( \rho \in T \), and zero initial conditions. Integrating Eq. 18 along this solution from \( t = 0 \) to \( t = T \) yields:

\[ V(x(T)) + \int_0^T d(t)^T d(t) dt + \frac{1}{\gamma} \int_0^T e(t)^T e(t) dt \leq \int_0^T d(t)^T d(t) dt. \]

It follows from \( \lambda_k \geq 0 \) that \( \Delta \in \Delta_{1/\gamma} \). In addition, \( \Psi \) is a valid time domain IQC for \( \Delta \). Apply this time domain IQC along with \( \rho \geq 0 \) to conclude that \( ||e|| \leq \gamma ||d|| \). Hence \( H_\rho \) achieves robust performance of level \( \gamma \).

The parameterized matrix inequality (Eq. 16) involves \( N \) IQCs. Note that left/right multiplying Eq. 16 by \( [x^T, w^T, d^T] \) and \( [x^T, w^T, d^T]^T \) does not yield a true dissipation inequality for two reasons. First, \( \Psi_k, M_k \) does not need to be a valid factorization and hence is not a valid time domain IQC. Second, the matrix \( P \) need not be positive definite and thus does not necessarily define a valid storage function. Lemma 1 addresses both issues.

Theorem 2. Assume \( F_u(H_\rho, \Delta) \) is well posed for all \( \Delta \in \Delta_{1/\gamma} \). Then \( H_\rho \) achieves robust performance of level \( \gamma \) if there exists a matrix \( P = P^T \in \mathbb{R}^{(n_u+n_v) \times (n_u+n_v)} \) and scalars \( \lambda_k \) such that \( (P, \lambda, \gamma) \) satisfies the parameterized matrix inequality in Eq. 16 for all \( \rho \in \mathcal{P} \) and \( \lambda_k \in \mathbb{R}_{>0} \) \( k = 2, \ldots, N \).

Proof. As described above, \( \Pi \) has a rescaled J-spectral factorization \( (\Psi, M) \). Define \( \tilde{P} := P - \begin{bmatrix} 0 & 0 \end{bmatrix} \) \( \geq 0 \) where \( X \) is the stabilizing ARE solution used to construct this factorization. By Lemma 1, \( \Pi \) satisfies Eq. 17. Define the storage function \( V = e^T P e \) \( \rho \in \mathcal{P} \) and \( \rho \in \mathcal{P} \).

4.2 Scaled System

This section constructs a scaled system that is used to link the analysis and synthesis steps in our robust synthesis algorithm. Consider the uncertain system \( F_u(H_\rho, \Delta) \). Theorem 2 provides a sufficient condition to upper bound the robust performance of \( H_\rho \). Recall that \( M_k := \text{diag}(\gamma^{-2T}I, I) \). Thus partitioning \( z_k := \begin{bmatrix} v_k & 0 \end{bmatrix} \) simplifies the dissipation inequality (Eq. 18) to \( V \leq (d^T d - \gamma^{-2} e^T e) + (w_1^T w_1 - \gamma^{-2} w_2^T w_2) \).

This has the form of a dissipation inequality used to prove a (nominal) LPV system from inputs \( (w_1, d) \) to outputs \( (v_1, e) \) has \( L_2 \) gain \( \gamma \). A scaled system is constructed based on this insight. First, rewrite the extended system of \( H_\rho \) and \( \Psi \) (Eq. 15) by partitioning \( z_k := \begin{bmatrix} v_k \end{bmatrix} \):

\[ \begin{bmatrix} \dot{x} \\ v_\lambda \\ w_\lambda \\ e \end{bmatrix} = \begin{bmatrix} A(\rho) & B_w(\rho) & B_d(\rho) \\ C_{v_\lambda}(\rho) & D_{v_\lambda,w}(\rho) & D_{v_\lambda,d}(\rho) \\ C_{w_\lambda}(\rho) & D_{w_\lambda,w}(\rho) & D_{w_\lambda,d}(\rho) \\ C_e(\rho) & D_{e,w}(\rho) & D_{e,d}(\rho) \end{bmatrix} \begin{bmatrix} x \\ w \\ d \end{bmatrix} \]

Assume that \( D_{w_\lambda, w} \) is nonsingular \( \forall \rho \in \mathcal{P} \). Then the output equation for \( v_\lambda \) can be rewritten as:

\[ w = D_{w_\lambda, w}^{-1} (w_\lambda - C_{w_\lambda,x} x - D_{w_\lambda,d} d) \]

Use this relation to substitute for \( w \) in Eq. 19. This gives the following “scaled” system:

\[ \begin{bmatrix} \dot{x} \\ v_\lambda \\ w_\lambda \\ e \end{bmatrix} = \begin{bmatrix} A(\rho) & B_w(\rho) & B_d(\rho) \\ C_{v_\lambda}(\rho) & D_{v_\lambda,w}(\rho) & D_{v_\lambda,d}(\rho) \\ C_{w_\lambda}(\rho) & D_{w_\lambda,w}(\rho) & D_{w_\lambda,d}(\rho) \\ C_e(\rho) & D_{e,w}(\rho) & D_{e,d}(\rho) \end{bmatrix} \begin{bmatrix} x \\ w_\lambda \\ d \end{bmatrix} \]

where \( T(\rho) \) is defined as:

\[ T(\rho) := \begin{bmatrix} -D_{w_\lambda, w}^{-1}(\rho) C_{w_\lambda}(\rho) & D_{w_\lambda, w}^{-1}(\rho) & 0 & 0 \\ 0 & 0 & -D_{w_\lambda, d}(\rho) & D_{w_\lambda,d}(\rho) \end{bmatrix} \]

The next lemma gives a formal statement connecting robust performance of the extended system to nominal performance of this scaled system.
Lemma 2. Let $\tilde{P} \geq 0$ and $\gamma > 0$ be given. The following statements are equivalent:

1. $(\tilde{P}, \gamma)$ satisfy the robust performance LMI (Eq. 17).
2. $\mathcal{D}_{w_1w}$ is nonsingular. Let $(A_{x\lambda}, B_{x\lambda}, C_{x\lambda}, D_{x\lambda})$ denote the state-space representation of the scaled system formed from $H_\rho$ and $\Psi_\lambda$ (Eq. 20). $(\tilde{P}, \gamma)$ satisfy the induced $L_2$ gain LMI (Eq. 2) associated with the scaled system $\forall \rho \in \mathcal{P}$:

$$\left[ \begin{array}{cc} \dot{\rho}A_{x\lambda} + A_{x\lambda}^T \tilde{P} B_{x\lambda} & P B_{x\lambda} \\ B_{x\lambda}^T P & -I \end{array} \right] + \frac{1}{\gamma^2} \left[ \begin{array}{c} C_{x\lambda} \\ D_{x\lambda} \end{array} \right] \leq 0 \quad (21)$$

Proof. $(1 \Rightarrow 2)$ Assume statement 1 holds. The $(2,2)$ block of Eq. 17 implies $\mathcal{D}_{w_1w}^T \mathcal{D}_{w_1w} > \gamma^{-2}(\mathcal{D}_{v_1w}^T \mathcal{D}_{v_1w} + \mathcal{D}_{w_1w}^T \mathcal{D}_{w_1w}) \geq 0$ and hence $\mathcal{D}_{w_1w}$ is nonsingular. Next, note that $T$ is nonsingular $\forall \rho \in \mathcal{P}$. Multiply Eq. 17 on the left/right by $T^T/T$ to demonstrate that Eq. 21 holds. $(2 \Rightarrow 1)$ follows by the inverse transformation.

The lemma states that $H_\rho$ satisfies the robust performance condition if and only if the scaled system satisfies the nominal performance condition. This scaled system is a complicated function of the state matrices of $H_\rho$ and $\Psi_\lambda$. This is an issue because the robust synthesis algorithm requires the use of this result with the closed-loop, $H_\rho := \mathcal{F}(G_\rho, K_\rho)$. In fact, the scaled system has a simpler construction. It is formed by inverting the input/output channel associated with $w$ to $w_\lambda$. This channel only involves the filter $\Psi_\lambda$ which can be expressed as:

$$\begin{bmatrix} \dot{x}_\psi \\ v_\lambda \\ w_\lambda \end{bmatrix} = \begin{bmatrix} \bar{A} & \gamma^{-1} \bar{B}_w & \bar{B}_w \\ \bar{C}_{v\lambda} & \bar{D}_{v\lambda} & \bar{D}_{v\lambda} \\ \bar{C}_{w\lambda} & \bar{D}_{w\lambda} & \bar{D}_{w\lambda} \end{bmatrix} \begin{bmatrix} x_v \\ v \\ w \end{bmatrix} \quad (22)$$

The condition $(\Pi_\lambda)^{\dagger}$ is sufficient to ensure that $\mathcal{D}_{w_1w}$ is nonsingular. Then $w$ can be solved as: $w = \bar{D}_{w_1w}^{-1}(w_\lambda - \bar{C}_{w\lambda}x_v - \bar{D}_{w\lambda}v)$. In this case, let $\Psi_\lambda^\dagger$ denote the filter from $(v_\lambda, w_\lambda)$ to $(v, w)$ obtained by inverting the $w$ to $w_\lambda$ channel of $\Psi_\lambda$:

$$\begin{bmatrix} \dot{x}_\psi \\ v_\lambda \\ w \end{bmatrix} = \begin{bmatrix} \bar{A}(\rho) & \gamma^{-1} \bar{B}_w(\rho) & \bar{B}_w(\rho) \\ \bar{C}_{v\lambda}(\rho) & \bar{D}_{v\lambda}(\rho) & \bar{D}_{v\lambda}(\rho) \\ \bar{C}_{w\lambda}(\rho) & \bar{D}_{w\lambda}(\rho) & \bar{D}_{w\lambda}(\rho) \end{bmatrix} \begin{bmatrix} x_v \\ v \\ w \end{bmatrix} \quad (23)$$

where $\bar{\mathcal{T}}(\rho)$ is defined as:

$$\bar{\mathcal{T}}(\rho) := \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I \end{bmatrix}$$

The next lemma provides an alternative, but equivalent, construction for the scaled system.

Lemma 3. Assume $\mathcal{D}_{w_1w}$ is nonsingular so that $\Psi_\lambda^\dagger$ as defined in Eq. 23 is well-defined. Moreover, assume $\mathcal{D}_{w_1w}$ is nonsingular $\forall \rho \in \mathcal{P}$ so that the scaled system formed from $H_\rho$ and $\Psi_\lambda$ (Eq. 20) is well-posed. Then the scaled system is equivalent to the LFT interconnection of $H_\rho$ and $\Psi_\lambda^\dagger$ as shown in Fig. 5.

4.3 Main Theorem

The main technical result for the proposed algorithm is that the iteration is well posed at each step and the robust performance is non-increasing at each iteration. Thus the robust performance will converge and the iteration will terminate. As with DK synthesis, there are no guarantees that the iteration will lead to a local optima let alone a global optima. However, it is a useful heuristic that enables robust synthesis to extend from LTI to LPV systems. This result is now stated.

Theorem 3. The iteration is well-posed at each step and the iteration is non-increasing, i.e. $\gamma(i) \leq \gamma(i-1)$ for $i = 1, 2, \ldots$

Proof. The initial iteration $i = 1$ differs slightly from the subsequent ones. Specifically, $\lambda(0) = [1, 0, \ldots, 0]$ yields $\Pi_\lambda(0) = \Pi_1$ in Step 6. The definition of $\Pi_1$ implies that it has a $J$-factorization with $\Psi_1 := I_{n+1+n_w}$ and $M_1 := \text{diag}(I_{n+1}, -I_{n_w})$ in Step 7. Since no rescaling is used on the first iteration, the scaled system in Step 7 is simply $G_\rho^s = G_\rho$. The synthesis step 8 is then performed with no modifications and yields a controller $K_1(1)$ that stabilizes $G_\rho$ and achieves a closed-loop gain $\nu(1) < \infty$. The analysis step then achieves a robust performance $\gamma(1) < \infty$ because $H_\rho$ is stable. Thus the first iteration is well-posed and achieves $\gamma(1) < \gamma(0) = +\infty$.

Subsequent iterations ($i > 1$) begin with the iteration count update (Step 4) and performance scaling definition (Step 5). Next the combined multiplier $\Pi_\lambda$ is constructed. It has a $J$-spectral factorization. In addition, $(\Pi_\lambda)^{\dagger}(+\infty) < 0$ implies that $\mathcal{D}_{w_1w}$ must be nonsingular. Hence by Lemma 3, the construction of $\Psi_\lambda^\dagger$ in Step 3 is well-defined. The analysis step from the previous iteration shows that there exists $(P(i-1), \lambda(i-1), \gamma(i-1))$ satisfying Eq. 16. By Lemma 1, this implies the existence of $\tilde{P}(i-1) \geq 0$ that, along with $(\lambda(i-1), \gamma(i-1))$, satisfies Eq. 17. Next, Lemma 2 states that feasibility of Eq. 17 implies that the scaled closed-loop of $H_\rho := \mathcal{F}(G_\rho, K_i(i-1))$ and $\Psi_\lambda$ is well-posed and has induced gain $\leq \gamma(i-1)$. By Lemma 3, this scaled system can be represented by the LFT of $H_\rho$ and $\Psi_\lambda^\dagger$.
\( \Psi_i \) as shown in Fig. 5. Removing the controller yields the scaled open-loop plant. Thus the construction of the scaled system in Step 7 is well-defined. Finally, the synthesis in Step 8 optimizes over all stabilizing controllers. Hence the new controller \( K_p(i) \) must yield better nominal performance than \( K_p(i-1) \): \( \nu(i) \leq \gamma(i-1) \). Thus \( K_p(i) \) must satisfy the nominal performance LMI in Eq. 21 with the slightly larger cost of \( \gamma := \gamma(i-1) \). Lemmas 2 and 1 can be used backward to the analysis condition in Step 9. Specifically, the closed-loop with \( K_p(i) \) satisfies the analysis condition in Step 9 with \( \gamma(i-1), \lambda(i-1) \) and \( P(i-1) \). Step 9 involves optimizing over all feasible \( \lambda \) and \( P \). This yields a robust performance cost no greater than the previous step \( \gamma(i) \leq \gamma(i-1) \).

5 Numerical Example

An example is used to demonstrate the applicability of the proposed robust synthesis algorithm. The example is based on an example that appears in Veenman and Scherer (2014) to test an alternative IQC synthesis algorithm for LTI systems. Here the example is extended to include plant dynamics described by an LPV system. As shown in Fig. 6, the nominal plant dynamics are given by the following LPV system \( F_{\rho} \):

\[
\begin{align*}
\dot{x}(t) &= \left( \frac{1}{71 + 2\rho I_2} \right) x(t) + \left( \frac{1}{71 + 2\rho I_2} \right) u(t) \\
y(t) &= \left[ \begin{array}{cc}
87.2 + 0.2s & -87.2 + 0.2s \\
107.4 + 0.2s & -110.4 + 0.2s
\end{array} \right] x(t)
\end{align*}
\]

(24)

(25)

\( F_{\rho} \) depends on a single scheduling parameter \( \rho \in [1, 3] \). The objective is to synthesize a robust controller \( K_{\text{rob}} \) that offers good tracking performance at low frequencies while penalizing control input at high frequencies. These objectives are specified via the weights \( W_e = \frac{3 + 10}{3 + 10} I_2 \) and \( W_u = \frac{4 + 10}{4 + 10} I_2 \) on the error \( e \) and control input \( u \), respectively. The controller should also be robust to the uncertainty \( \Delta \) which is a block diagonal nonlinear perturbation, i.e., \( \Delta := \text{diag}(\Delta_1, \Delta_2) \). Each block of \( \Delta \) is a (scalar) dead zone operator \( \Delta_i(v_i) \) defined by:

\[
w_i = \Delta_i(v_i) :=
\begin{cases}
v_i - b_i, & v_i > b_i \\
0, & v_i \in [-b_i, b_i] \\
v_i + b_i, & v_i < -b_i
\end{cases}
\]

(26)

where \( b_i = 0.05 \ (i = 1, 2) \). The uncertainty weight is defined as \( W_\Delta := \text{diag}(0.6, 0.3) \).

Three IQCs are chosen to describe each dead zone \( \Delta_i \). The first is \( \Pi_0 = \text{diag}(1, -1) \). The second one \( \Pi_6 = \left[ \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \) is used to model the [0, 1] sector bound (Megretski and Rantzer, 1997) on the dead zone. The last IQC \( \Pi_5 = \left[ \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] + H(s) \) with \( H(s) = \frac{1}{s^2+1} \) corresponds to a Zames-Falb multiplier. This is used to model the monotonic odd nonlinearity (Megretski and Rantzer, 1997). These multipliers can be combined to obtain the following multiplier for \( \Delta \):

\[
\begin{pmatrix}
\Pi_0 & \Pi_5 \\
\Pi_0 & \Pi_5
\end{pmatrix} := \begin{pmatrix}
\Pi_{11} & 0 & 0 \\
0 & \Pi_{12} & 0 \\
0 & 0 & \Pi_{12}
\end{pmatrix}
\]

(27)

Five extended IQCs (Eq. 27) are constructed to model \( \Delta \): \( \Pi_1 := (\Pi_0, \Pi_0) \), \( \Pi_2 := (\Pi_0, \Pi_0) \), \( \Pi_3 := (\Pi_0, \Pi_0) \), \( \Pi_4 := (\Pi_0, \Pi_0) \) and \( \Pi_5 := (\Pi_0, \Pi_0) \). It is easy to check that \( \{\Pi_i\}_{i=1}^5 \) satisfy Assumptions 2, 3 and 4 in Section 3.1.

To apply the proposed algorithm, \( F_{\rho} \) is approximated with 5 points spaced equally in the parameter range [1, 3]. After 3 iterations (46.89 s), robust performance with the designed controller \( K_{\text{rob}} \) converges to 0.96 using a stopping criteria \( \epsilon_{\text{tol}} = 0.05 \). As a comparison, a nominal LPV controller \( K_{\text{nom}} \) is designed for the system without uncertainty (\( \Delta = 0 \)). The induced \( L_2 \) norm of the nominal system using \( K_{\text{nom}} \) and \( K_{\text{rob}} \) is given by 0.42 and 0.56, respectively. Next, the robust performance was assessed using the matrix inequality condition in Section 4.1. This yields \( 3.03 \) and \( 0.96 \) for \( K_{\text{nom}} \) and \( K_{\text{rob}} \), respectively. The gap in robust performance between the two controllers is also illustrated by a time domain step response simulation (Fig. 7). In the simulation, unit step signals are injected into both channels of \( d \) simultaneously at \( t = 10 \) s and the parameter trajectory is given by \( \rho(t) = \sin(0.05 t) + 2 \). The responses of \( y_2 \) and \( y_2 \) are shown in Fig. 7. It is seen that \( K_{\text{nom}} \) performs well (solid blue curve) when there is no uncertainty. However, it degrades dramatically (dash-dot red curve) when the uncertainty is added. In contrast, \( K_{\text{rob}} \) maintains good tracking and steady state error (dash green curve) even in the presence of the uncertainty.

6 CONCLUSION

This paper described a robust synthesis algorithm for a class of uncertain LPV systems. The proposed algorithm involves a coordinate-wise iteration between an LPV synthesis step and an IQC analysis step. It was shown that the closed-loop robust performance is a non-increasing function of the iteration number. The effectiveness of this method was shown on a simple numerical example. Future work will consider refinements of the proposed algorithm including a more efficient parameterization of the IQC multipliers. In addition, the algorithm will be applied to design a robust LPV controller for a more realistic system.
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References


A Proof of Lemma 1

Proof. (⇒) Assume P = P T satisfies Eq. 16. The output zk from Ψ1/γ is a linear function of (xψ, v, w) as defined in Eq. 10: zk = [ ˜C ik γ−1 ˜D ik γ−1 ˜D kw ] [ xψ v w ] T . These variables (xψ, v, w) can, in turn, be expressed in terms of the extended system state and inputs (x, w, d) as:

\[ \begin{bmatrix} [x_{\psi} v w] \\ \end{bmatrix} = L(p) \begin{bmatrix} [x d] \\ [w d] \\ \end{bmatrix} \]  

(A.1)

where L(p) is defined as:

\[ L(p) := \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \end{bmatrix} \begin{bmatrix} C_p(p) & 0 \\ \end{bmatrix} \begin{bmatrix} D_{w}(p) & D_{w}(p) \\ 0 & 0 \\ \end{bmatrix} \begin{bmatrix} [x_{\psi} v w] \\ w d \\ \end{bmatrix} \]
Thus, using the extended system state matrices, the second term of the matrix inequality in Eq. 16 can be rewritten as:

\[
\sum_{k=1}^{N} \lambda_k \begin{bmatrix} c_{zk}^T & D_{zk}^T \end{bmatrix} M_k \begin{bmatrix} c_{zk} & D_{zk} \end{bmatrix} = L(\rho)^T \left( \begin{bmatrix} \tilde{Q}_\lambda & \tilde{S}_\lambda \end{bmatrix} \right) L(\rho) \]

where \( \tilde{Q}_\lambda, \tilde{S}_\lambda, \) and \( \tilde{R}_\lambda \) are defined in Eq. 13. Substitute for \( \tilde{Q}_\lambda \) using the ARE:

\[
\tilde{A}^T X + X \tilde{A} - (X \tilde{B} + \tilde{S}_\lambda)^T \tilde{R}_\lambda^{-1} (X \tilde{B} + \tilde{S}_\lambda)^T + \tilde{Q}_\lambda = 0 \tag{A.2}
\]

Rearrange terms in the matrix inequality to show that \( \tilde{P} := P - [0 \ 0 \ X] \) satisfies Eq. 17.

This direction of the proof is completed by showing that \( \tilde{P} \geq 0 \). Define \( V(x_0) := x_0^T \tilde{P} x_0 \) and the cost functional \( V^*(x_0) \):

\[
V^*(x_0) := \sup_{w \in L^2_v[0,\infty)} \int_0^\infty z_\lambda(t)^T M_\lambda z_\lambda(t) \ dt \tag{A.3}
\]

subject to:

\[
\dot{x} = \mathcal{A}(\rho)x + \mathcal{B}(\rho)w, \quad x(0) = x_0 \\
\dot{z}_\lambda = \mathcal{C}_z(\rho)x + \mathcal{D}_z w(\rho)w
\]

The disturbance input of the extended system is neglected (\( d = 0 \)) in this linear quadratic optimization. Note that the extended system is stable since \( \mathcal{H}_\rho \) is stable (by assumption), \( \Psi_\lambda \) is stable (by construction), and \( \Psi_\lambda \) is connected in an open loop fashion to \( \mathcal{H}_\rho \). First we show that \( V(x_0) \geq V^*(x_0) \) for all \( x_0 \in \mathbb{R}^{n+nv} \). This follows along the lines of Theorems 2 and 3 in Willems (1971) and hence the proof is only sketched. Let \( x(t), \ z_\lambda(t) \) be the resulting solutions of the extended system of \( \mathcal{H}_\rho \) and \( \Psi_\lambda \) for a given input \( w \in L^2_v[0,\infty) \), admissible trajectory \( \rho \in T \), and initial condition \( x_0 \in \mathbb{R}^{n+nv} \) assuming \( d = 0 \). Multiply the matrix inequality in Eq. 17 on the left/right by \( \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}^T \) and \( \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \) to show:

\[
\dot{V}(x(t)) + z_\lambda(t)^T M_\lambda z_\lambda(t) \leq 0. \int_0^T z_\lambda(t)^T M_\lambda z_\lambda(t) \ dt \leq V(x_0) \tag{A.4}
\]

\[
\lim_{T \to \infty} x(T) = 0 \text{ for any } w \in L^2_v[0,\infty) \text{ because the extended system is stable}. \text{ Maximizing the left side of Eq. A.4 over } w \in L^2_v[0,\infty) \text{ for } T = \infty \text{ thus yields} \ V(x_0) \geq V^*(x_0) \]

Next, consider the max/min game defined for the rescaled J-spectral factorization (\( \Psi_\lambda, M_\lambda \)):

\[
J(x_\psi) := \sup_{w \in L^2_v[0,\infty)} \inf_{v \in L^2_v[0,\infty)} \int_0^\infty z_\lambda(t)^T M_\lambda z_\lambda(t) \ dt \tag{A.5}
\]

subject to:

\[
x_\psi = \tilde{A} x_\psi + \tilde{B} [w], \quad x_\psi(0) = x_{\psi 0} \\
z = \tilde{C}_z x_\psi + \tilde{D}_z [w]
\]

where \( \tilde{D}_{z \lambda} := [\tilde{D}_{z \lambda v}, \tilde{D}_{z \lambda w}] \). This max/min game is connected to the quadratic optimization defined in Eq. A.3. Specifically, restricting \( v \) in the max/min game to be the output of \( H_\rho \) generated by \( w \in L_2, d = 0 \), and \( x_H(0) = x_{H0} \) yields the quadratic optimization in Eq. A.3. This specific choice of \( v \) yields a value that is no lower than the infimum over all possible \( v \in L_2 \). Hence the max/min game yields the bound \( J(x_\psi) \leq V^*(x_0) \). By Theorem 4 of Seiler (2015), the cost of this max/min game is \( J(x_\psi) = 0 \). Putting these results together yields the following inequality 0 = \( J(x_\psi) \) \( \leq V^*(x_0) \) \( \leq V(x_0) := x_0^T \tilde{P} x_0 \). This holds for any \( x_0 \) and thus \( \tilde{P} \geq 0 \).