Gain Scheduling for Nonlinear Systems via Integral Quadratic Constraints

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Abstract—The paper considers a general approach for gain scheduling of Lipschitz continuous nonlinear systems. The approach is based on a linear parameter varying system (LPV) representation of the nonlinear dynamics along with integral quadratic constraints (IQC) to account for the linearization errors. Past results have shown that Jacobian linearization leads to hidden coupling terms in the controlled system. These terms arise due to neglecting the higher order terms of the Taylor series and due to the use of constant (frozen) values of the scheduling parameter. This paper proposes an LPV control synthesis method that accounts for these shortcomings. The higher order terms of the linearization are treated as a memoryless uncertainty whose input/output behavior is described by a parameter varying IQC. It is also shown that if the rate of the scheduling parameter is measurable then it can be treated as a known disturbance in the control synthesis step. A simple numerical example shows that the proposed control design approach leads to improved control performance.

I. INTRODUCTION

Gain scheduling is a common approach to nonlinear control design [1], [2], [3], [4]. The starting point for gain scheduling design is an LPV model of the nonlinear plant generally obtained by Jacobian linearization about a family of equilibrium (trim) points as given in Section II. A linear controller is designed at each trim point of the plant ensuring that the performance criteria are met locally. The nonlinear controller is constructed by interpolating between the linear controllers based on the scheduling parameter. Two main directions exist for LPV system representation, linear fractional transformation (LFT) based LPV systems [4], [5], [6] and ”grid-based” LPV systems [7], [8]. The former requires rational dependence on the parameters, but leads to more computationally tractable linear matrix inequality (LMI) conditions while the latter offers arbitrary dependence on the parameter. The paper follows the grid-based approach, but the results may be extended to LFT type LPV systems.

The main advantage of gain scheduling is that it applies well developed linear design tools to nonlinear problems. The induced L2 control design approach is given in Section III-A. On the other hand, a major limitation of gain scheduling is that the closed-loop system fulfills the stability and performance criteria only in the vicinity of the trim points. It was shown in [2], [9], [10], [11], [12], [13], [14] that hidden coupling terms can appear in the closed loop due to neglecting the higher order terms of the Taylor series in the linearization and due to variation in the scheduling parameter.

The aim of the paper is to propose a control synthesis method that accounts for these shortcomings of gain scheduling. The paper considers Lipschitz-continuous nonlinear systems. The higher order terms of the linearization of such systems can be treated as a memoryless uncertainty whose input/output signals are described by a parameter varying IQC. IQCs provide a general framework for robustness analysis [15], where the interconnection of a linear system and a perturbation is considered and the input/output behavior of the perturbation is bounded by an IQC in the frequency domain (Section III-B). The IQC framework is extended to the time domain based on the dissipation inequality in [6], [18] and parameter varying IQCs are introduced in [17], [18]. Hidden couplings arise in the linearization process due to the time variation of the scheduling parameter. This variation can be treated as a disturbance in the design (synthesis) model. In addition, the LPV controller can explicitly depend on the parameter rate of variation if it is measurable [7], [8]. This offers guarantees on stability and performance in the case of time-varying scheduling parameter(s). The proposed control design and a numerical example are given in Sections IV–V.

II. PROBLEM FORMULATION

A. Assumptions

Consider the following nonlinear system $G$:

\[
\begin{align*}
\dot{x}(t) &= f(x(t), d(t), u(t), \rho(t)) \\
\epsilon(t) &= h_1(x(t), d(t), u(t), \rho(t)) \\
y(t) &= h_2(x(t), d(t), u(t), \rho(t))
\end{align*}
\]

where $f$, $h_1$ and $h_2$ are differentiable functions. The signals are input $u(t) \in \mathbb{R}^{n_u}$, disturbance $d(t) \in \mathbb{R}^{n_d}$, measured output $y(t) \in \mathbb{R}^{n_y}$, performance output $\epsilon(t) \in \mathbb{R}^{n_e}$ and state variable $x(t) \in \mathbb{R}^{n_x}$. Finally, $\rho(t) \in \mathbb{R}^{n_\rho}$ is a measurable exogenous parameter vector, called the scheduling parameter. $\rho$ is assumed to be a continuously differentiable function and the admissible trajectories are restricted based on physical considerations to a known compact subset $\mathcal{P} \subset \mathbb{R}^{n_\rho}$. The rates of the parameter variation $\dot{\rho}$ are assumed to be bounded in some applications. The present paper investigates the unbounded rate case for simplicity. The results carry over to the rate bounded case with a more complex notation. The dependence on time $t$ is suppressed to shorten the notation.

Assumption 1: $f$, $h_1$ and $h_2$ are Lipschitz-continuous:

\[
\begin{align*}
&\|f(\alpha_1) - f(\alpha_2)\| \leq L_f \|\alpha_1 - \alpha_2\| \quad \forall \alpha_1, \alpha_2 \in \text{dom } f \\
&\|h_1(\alpha_1) - h_1(\alpha_2)\| \leq L_{h_1} \|\alpha_1 - \alpha_2\| \quad \forall \alpha_1, \alpha_2 \in \text{dom } h_1 \\
&\|h_2(\alpha_1) - h_2(\alpha_2)\| \leq L_{h_2} \|\alpha_1 - \alpha_2\| \quad \forall \alpha_1, \alpha_2 \in \text{dom } h_2
\end{align*}
\]

where $L_f, L_{h_1}, L_{h_2} \in \mathbb{R}^+_{\text{d}}$ are the Lipschitz constants for $f$, $h_1$ and $h_2$, respectively.
Assumption 2: There is a family of equilibrium points $(\bar{x}(\rho), \bar{d}(\rho), \bar{u}(\rho))$ such that

$$f(\bar{x}(\rho), \bar{d}(\rho), \bar{u}(\rho)) = 0, \quad \forall \rho \in \mathcal{P}$$  \hspace{1cm} (3)

The parameterized trim outputs are defined as

$$\bar{e}(\rho) = h_1(\bar{x}(\rho), \bar{d}(\rho), \bar{u}(\rho)), \quad \forall \rho \in \mathcal{P}$$
$$\bar{y}(\rho) = h_2(\bar{x}(\rho), \bar{d}(\rho), \bar{u}(\rho)), \quad \forall \rho \in \mathcal{P}$$  \hspace{1cm} (4)

The general control objective is to ensure that $x$ tracks $\bar{x}(\rho)$. Note that $\rho$ specifies the desired operating point and is effectively a reference command.

B. Jacobian Linearization of Nonlinear Systems

The nonlinear system $G$ given by (1) can be linearized about the equilibrium points via Jacobian linearization based on Taylor series expansion. Define the deviation variables as

$$\delta_x := x - \bar{x}(\rho), \quad \delta_u := u - \bar{u}(\rho), \quad \delta_x := e - \bar{e}(\rho)$$
$$\delta_y := y - \bar{y}(\rho), \quad \delta_d := d - \bar{d}(\rho)$$  \hspace{1cm} (5)

Differentiating the $\delta_x$ term of (5) results in

$$\dot{\delta}_x = \dot{x} - \dot{\bar{x}}(\rho) = f(x, d, u, \rho) - \dot{\bar{x}}(\rho)$$  \hspace{1cm} (6)

The Taylor series expansion of $f$, $h_1$ and $h_2$ about the equilibrium point yields

$$\dot{\delta}_x = \nabla_x f(\bar{x}(\rho)) \delta_x + \nabla_h f(\bar{x}(\rho)) \delta_u + \nabla_u f(\bar{x}(\rho)) \delta_u + \epsilon_f(\delta_x, \delta_u, \rho) - \dot{\bar{x}}(\rho)$$
$$\delta_e = \nabla_x h_1(\bar{x}(\rho)) \delta_x + \nabla_u h_1(\bar{x}(\rho)) \delta_u + \epsilon_{h_1}(\delta_x, \delta_u, \rho)$$
$$\delta_y = \nabla_x h_2(\bar{x}(\rho)) \delta_x + \nabla_u h_2(\bar{x}(\rho)) \delta_u + \epsilon_{h_2}(\delta_x, \delta_u, \rho)$$  \hspace{1cm} (7)

where the $|_0$ denotes evaluation at the trim point $(\bar{x}(\rho), \bar{d}(\rho), \bar{u}(\rho), \rho)$. Terms $\epsilon_f$, $\epsilon_{h_1}$ and $\epsilon_{h_2}$ represent the higher order terms of the Taylor series expansion. The term $\dot{\delta}_x (\rho)$ arises due to the time variation in $\rho$. The linearization is performed with respect to $(x, d, u)$ but the nonlinear dependence on $\rho$ is retained. Define $L(\rho) := -\nabla_x \bar{x}(\rho)$. The linearization about the family of trim points becomes

$$\dot{\delta}_x = A(\rho)\delta_x + B_d(\rho)\delta_d + B_u(\rho)\delta_u + L(\rho)\delta_x + \epsilon_f(\delta_x, \delta_u, \rho)$$
$$\delta_e = C_e(\rho)\delta_x + D_{e,d}(\rho)\delta_d + D_{e,u}(\rho)\delta_u + \epsilon_e(\delta_x, \delta_d, \delta_u, \rho)$$
$$\delta_y = C_y(\rho)\delta_x + D_{u,d}(\rho)\delta_d + D_{u,u}(\rho)\delta_u + \epsilon_u(\delta_x, \delta_d, \delta_u, \rho)$$  \hspace{1cm} (8)

where the parameter-dependent state matrices are given by the gradients appearing in (7), e.g. $A(\rho) := \nabla_x f(\bar{x}(\rho))$. The LPV system is obtained by assuming that $\epsilon_f, \epsilon_{h_1}, \epsilon_{h_2} \approx 0$. In addition, it is typically assumed that the parameter variation is sufficiently slow, thus $\dot{\rho} \approx 0$. Under these assumptions, the LPV system $G_{\rho}$ is given by

$$\dot{\delta}_x = A(\rho)\delta_x + B_d(\rho)\delta_d + B_u(\rho)\delta_u$$
$$\delta_e = C_e(\rho)\delta_x + D_{e,d}(\rho)\delta_d + D_{e,u}(\rho)\delta_u$$
$$\delta_y = C_y(\rho)\delta_x + D_{u,d}(\rho)\delta_d + D_{u,u}(\rho)\delta_u$$  \hspace{1cm} (9)

The goal of the paper is to propose an LPV control synthesis method, which addresses these shortcomings of the Jacobian linearization. The terms $\epsilon_{f}$, $\epsilon_{h_1}$ and $\epsilon_{h_2}$ are treated as a memoryless uncertainty satisfying a parameter varying IQC. The term $L(\rho)$ is treated as a disturbance in the design (synthesis) model.

III. BACKGROUND

This section reviews existing material on LPV systems and IQCs.

A. Induced $L_2$ Control of LPV Systems

Consider an LPV system $G_{\rho}$ obtained via Jacobian linearization of the nonlinear system $G$,

$$\begin{bmatrix} \dot{x} \\ e \\ y \end{bmatrix} = \begin{bmatrix} A(\rho) & B_d(\rho) & B_u(\rho) \\ C_e(\rho) & D_{e,d}(\rho) & D_{e,u}(\rho) \\ C_y(\rho) & D_{u,d}(\rho) & D_{u,u}(\rho) \end{bmatrix} \begin{bmatrix} x \\ d \\ u \end{bmatrix}$$  \hspace{1cm} (10)

The $\delta$ notation that appears in (9) for the (linearized) deviation variables is dropped here in order to simplify the notation. Let $K_{\rho}$ be an LPV controller of the form:

$$\begin{bmatrix} \dot{x}_K \\ u \end{bmatrix} = \begin{bmatrix} A_K(\rho) & B_K(\rho) \\ C_K(\rho) & D_{u,cl}(\rho) \end{bmatrix} \begin{bmatrix} x_K \\ y \end{bmatrix}$$  \hspace{1cm} (11)

The controller $K_{\rho}$ generates the control input $u$ with a linear dependence on the measurement $y$ but an arbitrary dependence on the scheduling parameter $\rho$. A lower LFT $\mathcal{F}_I(G_{\rho}, K_{\rho})$ defines the closed-loop interconnection of $G_{\rho}$ and $K_{\rho}$ (see Fig. 1.a). The performance of $\mathcal{F}_I(G_{\rho}, K_{\rho})$ can be specified in terms of the induced $L_2$ gain from $d$ to $e$ over all allowable parameter trajectories as

$$\| \mathcal{F}_I(G_{\rho}, K_{\rho}) \| = \sup_{d \neq 0, 0 \leq t \leq T} \| \| e \| \|$$  \hspace{1cm} (12)

where $x_{cl}$ denotes the closed loop state variables. The objective is to synthesize a controller $K_{\rho}$ to minimize the closed-loop induced $L_2$ gain from $d$ to $e$. The following theorem gives the sufficient condition to upper bound the induced $L_2$ gain of $\mathcal{F}_I(G_{\rho}, K_{\rho})$.

Theorem 1: ([7], [8]): The interconnection $\mathcal{F}_I(G_{\rho}, K_{\rho})$ is exponentially stable and $\| \mathcal{F}_I(G_{\rho}, K_{\rho}) \| \leq \gamma$ if there exists a matrix $P = P^T \in \mathbb{R}^{n_x \times n_x}$ such that $P \geq 0$ and $\forall \rho \in \mathcal{P}$

$$\begin{bmatrix} PA_d + \frac{A^T_d P}{2} & PB_{cl} \\ B_{cl}^T P & -I \end{bmatrix} + \frac{1}{2} \begin{bmatrix} C_{e,cl}^T & C_{u,cl} \end{bmatrix} \frac{D_{e,cl}}{2} \begin{bmatrix} C_{e,cl} & D_{e,cl} \end{bmatrix} < 0$$  \hspace{1cm} (13)

where subscript $cl$ stands for closed loop. The dependence of the state matrices on $\rho$ has been omitted in (13).

Proof: The proof is based on a dissipation inequality satisfied by the storage function $V : \mathbb{R}^{n_{x,cl}} \times \mathbb{R}^{n_{x,cl}} \to \mathbb{R}^+$ given as $V(x_{cl}) := x_{cl}^T P x_{cl}$. Multiplying (13) on the left/right by $[x_{cl}^T, d^T]$ and $[x_{cl}^T, d^T]^\top$ gives

$$\dot{V} \leq d^T d - \gamma^{-2} e^T e$$  \hspace{1cm} (14)

The dissipation inequality (14) can be integrated with the initial condition $x_{cl}(0) = 0$, which yields $\| e \| \leq \gamma \| d \|$. This analysis theorem forms the basis for the induced $L_2$ norm controller synthesis of [7], [8], achieved by solving bounded-real type LMI conditions that are sufficient to upper bound the gain of an LPV system. The LMI conditions and the controller reconstruction steps are given in [7], [8].

B. Robustness Analysis of LPV Systems via Integral Quadratic Constraints

IQC provides a framework for robustness analysis [15]. The IQC specifies constraint on the input/output signals of the perturbation.

Definition 1: Let $M$ be a symmetric matrix, i.e. $M = M^T \in \mathbb{R}^{n_x \times n_x}$ and $\Psi$ a stable linear system, i.e. $\Psi \in \mathbb{RH}^{n_x \times (n_x + n_u)}$. Operator $\Delta : L_{2e}^{n_e} \to L_{2e}^{n_{w,\Psi}}$ satisfies IQC defined by $(M, \Psi)$ if the following inequality holds for all $v \in L_{2e}^{n_v}[0, \infty)$, $w = \Delta(v)$ and $T \geq 0$:

$$\int_0^T z^T M z dt \geq 0$$  \hspace{1cm} (15)
where $z$ is the output of the linear system $\Psi$ with inputs $(v, w)$ and zero initial conditions.

The notation $\Delta \in IQC(\Psi, M)$ is applied if $\Delta$ satisfies IQC defined by $\Psi, M$. Fig. 1.b shows a graphic interpretation of the IQC, where the input and output signals of $\Delta$ are filtered through $\Psi$. There is a wide class of IQCs available for various uncertainties or nonlinearities. The remainder of the section focuses on IQCs for a memoryless operator $\Delta$ based on time varying sector bounds. The input/output behavior of $\Delta$ is described by $w = \Delta(v, \rho)$ where signals $v$ and $w$ are assumed to satisfy the following condition:

$$v^T S(\rho)v - w^T w \geq 0, \forall v \in \mathbb{R}^{n_v}, w \in \mathbb{R}^{n_w}, \rho \in \mathcal{P}$$  \hspace{1cm} (16)

where $S(\rho)$ is a parameter dependent diagonal matrix that scales signal $v$. The uncertainty $\Delta$ therefore satisfies the quadratic constraint (QC)

$$M(\rho) = \begin{bmatrix} S(\rho)^T S(\rho)I_{n_v} & 0 \\ 0 & -I_{n_w} \end{bmatrix}$$ \hspace{1cm} (17)

where $M(\rho)$ is defined as

$$M(\rho) := \begin{bmatrix} S(\rho)^T S(\rho)I_{n_v} & 0 \\ 0 & -I_{n_w} \end{bmatrix}$$ \hspace{1cm} (18)

Selecting $\Psi = I_{n_v+n_w}$, therefore $z = [v^T \ w^T]^T$, and integrating (17) implies $\Delta \in IQC(I, M(\rho))$.

The uncertain LPV system denoted by upper LFT as $\mathcal{F}_u(H_\rho, \Delta)$ is defined by the interconnection of an LPV system $H_\rho$ and uncertainty $\Delta$. $H_\rho$ is defined as

$$\begin{bmatrix} \dot{x} \\ v \\ e \end{bmatrix} = \begin{bmatrix} A(\rho) & B_w(\rho) & B_d(\rho) \\ C_v(\rho) & D_{vw}(\rho) & D_{wd}(\rho) \\ C_e(\rho) & D_{ew}(\rho) & D_{ed}(\rho) \end{bmatrix} \begin{bmatrix} x \\ w \\ d \end{bmatrix}$$ \hspace{1cm} (19)

The worst-case $L_2$ gain of $\mathcal{F}_u(H_\rho, \Delta)$ can be defined as

$$\gamma := \sup_{\Delta \in IQC(I, M(\rho)),\rho \in \mathcal{P}} \| \mathcal{F}_u(H_\rho, \Delta) \|$$ \hspace{1cm} (20)

An upper bound to the worst-case $L_2$ gain $\gamma$ can be defined as a dissipation inequality based on equations (17) and (19) in the form of an LMI [17], [18].

**Theorem 2:** Assume $\mathcal{F}_u(H_\rho, \Delta)$ is well posed for all $\Delta \in IQC(I, M(\rho))$. Then $\| \mathcal{F}_u(H_\rho, \Delta) \| \leq \gamma$ if there exists matrix $P = P^T \in R_{n_v \times n_v}$ and a scalar $\lambda \geq 0$ such that $P \geq 0$ and $\forall \rho \in \mathcal{P}$

$$\begin{bmatrix} PA + A^T P & PB_w & PB_d \\ D_w^TP & 0 & 0 \\ D_d^TP & 0 & -I \end{bmatrix} + \frac{1}{\gamma^2} \begin{bmatrix} C_v^T \\ C_e^T \\ I \end{bmatrix} \begin{bmatrix} D_{vw}^T \\ D_{ew}^T \\ D_{vd}^T \end{bmatrix} < 0$$ \hspace{1cm} (21)

The dependence on $\rho$ has been omitted in (21).

**Proof:** The proof is based on defining the storage function $V : \mathbb{R}^{n_x \times n_x} \to \mathbb{R}^+$ by $V(x) := x^TPx$. Left and right multiply (21) by $[x^T, w^T, d^T]$ and $[x^T, w^T, d^T]^T$ to show that $V$ satisfies the dissipation inequality:

$$\lambda \begin{bmatrix} v^T \\ w^T \end{bmatrix} M \begin{bmatrix} v^T \\ w^T \end{bmatrix} + V \leq \gamma^2 d^T d - e^T e$$ \hspace{1cm} (22)

The dissipation inequality (22) can be integrated from $t = 0$ to $t = T$ with the initial condition $x(0) = 0$. The QC condition (17) along with $\lambda \geq 0$ and $P \geq 0$ imply $\|e\| \leq \gamma \|d\|$. Details of the proof are given in [16], [18].

The results of the section can be considered as a specific case of parameter varying IQCs, where uncertainty $\Delta$ satisfies a more strict QC. The theory of IQC is more general in principle [6], [15], [16], [17], which can contain dynamic, parameter varying filters and integral constraints.

**IV. THE PROPOSED CONTROL DESIGN**

Consider the LPV system $G_\rho$ given by (9), obtained by Jacobian linearization of $G$ in (1). $G_\rho$ is an approximation of $G$ since terms $\epsilon_f, \epsilon_{h_1}, \epsilon_{h_2}$ and $L(\rho)\hat{\rho}$ are considered negligible in the linearization step. The aim of this section is to propose an LPV control design method for $G_\rho$ based on [7], [8] that accounts for these neglected terms. These terms can be formulated as perturbations to system $G_\rho$ and can be sorted into two groups.

The goal is to treat the higher order terms $\epsilon_f, \epsilon_{h_1}$ and $\epsilon_{h_2}$ as a memoryless uncertainty $\Delta$ whose input/output signals satisfy a QC. Uncertainty $\Delta$ can be derived based on Assumption 1. Interconnection $\mathcal{F}_u(G_\rho, \Delta)$ allows IQC-based robustness analysis. Additionally, the aim is apply scalings to $G_\rho$ and $\Delta$ such that IQC (13) for the resulting interconnection. Therefore, the LPV control synthesis of [7], [8] accounts for terms $\epsilon_f, \epsilon_{h_1}$ and $\epsilon_{h_2}$ of the interconnection. The term $L(\rho)\hat{\rho}$ can be treated as an additional disturbance or it can be incorporated as an input to the controller in the LPV design in case $\hat{\rho}$ is measurable. By accounting for these terms, the proposed control synthesis method gives an upper bound for the induced $L_2$ gain from input $d$ to output $e$ for interconnection of the resulting LPV controller and the original nonlinear system $G$.

**A. Quadratic Constraints for the Higher Order Terms of Taylor Series Expansion**

The first goal of this section is to derive uncertainty $\Delta$ that satisfies a QC and captures the terms $\epsilon_f, \epsilon_{h_1}, \epsilon_{h_2}$ from Taylor series expansion of the controller and the uncertain parameter dependent scalings. The second scaling accounts for the parameter dependent uncertainty $\Delta$. $\Delta$ is transformed to an identity matrix via parameter dependent scalings. The second scaling accounts for the parameter dependent uncertainty $\Delta$. $\Delta$ is transformed to an identity matrix via parameter dependent scalings. The second goal is to apply scalings to $G_\rho$ and $\Delta$ such that LMI (21) becomes equivalent to LMI (13) for the resulting interconnection. Therefore, the LPV control synthesis of [7], [8] accounts for terms $\epsilon_f, \epsilon_{h_1}$ and $\epsilon_{h_2}$ of the interconnection. The term $L(\rho)\hat{\rho}$ can be treated as an additional disturbance or it can be incorporated as an input to the controller in the LPV design in case $\hat{\rho}$ is measurable. By accounting for these terms, the proposed control synthesis method gives an upper bound for the induced $L_2$ gain from input $d$ to output $e$ for interconnection of the resulting LPV controller and the original nonlinear system $G$.
also Lipschitz-continuous. The behavior of these terms can be captured by a memoryless uncertainty \( \Delta \). The input/output signals of uncertainty \( \Delta \) satisfy a QC.

**Proof:** Consider the first element of \( \epsilon_f \), denoted by \( \epsilon_{f_1} \), which can be expressed based on (8) as

\[
\epsilon_{f_1}(\delta_x, \delta_d, \delta_u, p) = f_1(x, d, u, p) - f_1(\bar{x}, \bar{d}, \bar{u}, p) - A_1(p)\delta_x - B_{d1}(p)\delta_d - B_{u1}(p)\delta_u
\]

where \( A_1(p), B_{d1}(p) \) and \( B_{u1}(p) \) denote the first rows of matrices \( A(p), B_d(p) \) and \( B_u(p) \) respectively. The norm of (23) satisfies the following inequality

\[
\|\epsilon_{f_1}(\delta_x, \delta_d, \delta_u, p)\| \leq \|f_1(\bar{x} + \delta_x, \bar{d} + \delta_d, \bar{u} + \delta_u, p) - f_1(\bar{x}, \bar{d}, \bar{u}, p)\| + \|A_1(p)\delta_x\| + \|B_{d1}(p)\delta_d\| + \|B_{u1}(p)\delta_u\|
\]

(24)

Substituting Lipschitz condition (2) into (24) results in

\[
\|\epsilon_{f_1}(\delta_x, \delta_d, \delta_u, p)\| \leq \|f_1(\bar{x} + \delta_x, \bar{d} + \delta_d, \bar{u} + \delta_u, p) - f_1(\bar{x}, \bar{d}, \bar{u}, p)\| + \|A_1(p)\|\delta_x\| + \|B_{d1}(p)\|\delta_d\| + \|B_{u1}(p)\|\delta_u\|
\]

(25)

The following inequality holds based on the Euclidean norm

\[
\|\epsilon_{f_1}(\delta_x, \delta_d, \delta_u, p)\|^2 \leq (L_f(1)) \|\delta_x\|^2 + (L_f(1)) \|\delta_d\|^2 + (L_f(1)) \|\delta_u\|^2
\]

(26)

Applying Jensen's inequality to (26) leads to

\[
\|\epsilon_{f_1}(\delta_x, \delta_d, \delta_u, p)\|^2 \leq 3((L_f(1)) \|\delta_x\|^2 + (L_f(1)) \|\delta_d\|^2 + (L_f(1)) \|\delta_u\|^2)
\]

(27)

Finally, the following inequality can be obtained for \( \epsilon_{f_1} \)

\[
\epsilon_{f_1}(\delta_x, \delta_d, \delta_u, p)^2 \leq a_{\epsilon_{f_1}}^2 \delta_x^2 + a_{\epsilon_{f_1}}^2 \delta_d^2 + a_{\epsilon_{f_1}}^2 \delta_u^2
\]

(28)

Let signals \( w_{\epsilon_{f_1}} \) and \( v_{\epsilon_{f_1}} \) be defined as \( w_{\epsilon_{f_1}} := \epsilon_{f_1} \) and \( v_{\epsilon_{f_1}} := \left[ \delta_x^2 \delta_d^2 \delta_u^2 \right]^T \). Then the following QC holds

\[
\begin{bmatrix}
\epsilon_{f_1} \\
\delta_x \\
\delta_d \\
\delta_u \\
\end{bmatrix}^T \begin{bmatrix}
\epsilon_{f_1} \\
\delta_x \\
\delta_d \\
\delta_u \\
\end{bmatrix} \geq 0
\]

(29)

where \( M_{\epsilon_{f_1}}(p) \) is given as

\[
M_{\epsilon_{f_1}}(p) := \begin{bmatrix}
a_{\epsilon_{f_1}}(p)I_{nx} & 0 & 0 & 0 \\
0 & a_{\epsilon_{f_1}}(p)I_{nd} & 0 & 0 \\
0 & 0 & a_{\epsilon_{f_1}}(p)I_{nu} & 0 \\
0 & 0 & 0 & -1 \\
\end{bmatrix}
\]

(30)

A memoryless uncertainty \( \Delta_{\epsilon_{f_1}} \) can be constructed whose input/output behavior is given by \( w_{\epsilon_{f_1}} = M_{\epsilon_{f_1}}(p) \epsilon_{f_1} \) and \( \Delta_{\epsilon_{f_1}} \in QC(\{I, M_{\epsilon_{f_1}}(p)\}) \). QC's can be constructed for each term of \( \epsilon_f, \epsilon_{h_1} \) and \( \epsilon_{h_2} \) in the same manner. A block diagonal \( \Delta \) can then be constructed as

\[
\Delta = \begin{bmatrix}
\Delta_{\epsilon_{f_1}} & \cdots & \\
& \Delta_{\epsilon_{h_1}} & \\
& & \Delta_{\epsilon_{h_2}} \\
\end{bmatrix}
\]

(31)

where input/output behavior of \( \Delta \in QC(I, M(p)) \) is given by \( w = \Delta(v, p) \) where \( v = \left[ v_{x_{f_1}}^T \cdots v_{h_{2n}}^T \right]^T \) and \( w = \left[ w_{x_{f_1}} \cdots w_{h_{2n}} \right]^T \).

![Fig. 2. Scaled system](image)

The first scaling accounts for the term \( M(p) \) of LMI (21). Signal \( v \) can be scaled in the following way

\[
v = S(p)v
\]

(33)

where \( S(p) \) is defined as

\[
S(p) := \begin{bmatrix}
a_{\epsilon_{f_1}}(p)I_{nx} & \cdots & a_{\epsilon_{h_{2n}}}(p)I_{ny} \\
\cdots & \cdots & \cdots \\
\end{bmatrix}
\]

(34)

\( \Delta \) can be constructed as \( w = \Delta(v, p) \). The input/output behavior of \( \Delta \) satisfies

\[
v^Tw - w^Tw \geq 0, \quad \forall v \in \mathbb{R}^{nx}, \ w \in \mathbb{R}^{nw}
\]

(35)

The uncertainty \( \tilde{\Delta} \) therefore satisfies the QC

\[
\begin{bmatrix}
\tilde{v}^T \\
\tilde{w}^T \\
\end{bmatrix} M \begin{bmatrix}
\tilde{v} \\
\tilde{w} \\
\end{bmatrix} \geq 0, \quad \forall \tilde{v} \in \mathbb{R}^{nx}, \ \tilde{w} \in \mathbb{R}^{nw}
\]

(36)
where $\bar{M}$ is defined as

$$\bar{M} := \begin{bmatrix} I_{n_0} & 0 \\ 0 & -I_{n_u} \end{bmatrix}$$

(37)

The second scaling accounts for the optimization over $\lambda$. The aim is to pull $\lambda$ out from LMI (21) and treat it as scalings to $G_\rho$ and $\bar{\Delta}$. This can be achieved by scaling $G_\rho$ and $\bar{\Delta}$ as

$$\tilde{G}_\rho = \begin{bmatrix} \sqrt{\lambda} I_{n_0} & 0 \\ 0 & I_{n_u+n_v} \end{bmatrix} G_\rho \begin{bmatrix} \sqrt{\lambda} I_{n_0} & 0 \\ 0 & I_{n_d+n_u} \end{bmatrix}$$

$$\bar{\Delta} = \sqrt{\lambda} \Delta \sqrt{\lambda}^{-1}$$

B. Rate of Variation of $\rho$ as an External Disturbance

The aim of this section is to include the effect of $L(\rho) \dot{\rho}$ into $G_\rho$, $\dot{\rho}$ can be treated as an additional disturbance signal. Therefore, signal $d$ can be extended as $d = [d^T \ \dot{\rho}^T]^T$. $G_\rho$ can be derived from $\tilde{G}_\rho$ by extending the input matrices

$$B_{\tilde{d}}(\rho) = [B_d(\rho) \ L(\rho)], \ D_{\tilde{d}}(\rho) = [D_{vd}(\rho) \ 0], \ D_{\tilde{d}}(\rho) = [D_{vd}(\rho) \ 0]$$

(39)

$\tilde{G}_\rho$ and $\bar{\Delta}$ form the basis of the following theorem.

Theorem 3: Let the interconnection of controller $K_\rho$ in the form of (11) and the nonlinear system $G$ of (1) be denoted by $T$. Assume that $F_u(G_\rho, \bar{\Delta})$ is well posed for all $\Delta \in QC(\bar{M})$. Then controller $K_\rho$ can be designed such that $\|T\| \leq \gamma$ if there exists $\gamma \leq 1$, matrix $P = PT \in \mathbb{R}^{n_{\text{vd}} \times n_{\text{vd}}}$ such that $P \geq 0$ and $\forall \rho \in \mathcal{P}$

$$[P \bar{A}_{\tilde{d}} + \bar{A}_{\tilde{d}}^T P \bar{B}_{\tilde{d}}] + \frac{1}{\gamma} \bar{C}_{\tilde{d}}^T [\bar{C}_{\tilde{d}} \ \bar{D}_{\tilde{d}}] < 0$$

(40)

where $\bar{A}_{\tilde{d}}, \bar{B}_{\tilde{d}} = [\bar{B}_{vd}, \bar{B}_{d}]$, $\bar{C}_{\tilde{d}} = [\bar{C}_{vd}, \bar{C}_{d}]^T$ and $\bar{D}_{\tilde{d}} = \begin{bmatrix} \bar{D}_{vd}\bar{d} \\ \bar{D}_{d}\bar{d} \end{bmatrix}$ are the state matrices for the closed loop lower LFT $F_L(G_\rho, K_\rho)$. The dependence of the state matrices on $\rho$ has been omitted in (40).

Proof: Note that LMI (40) can be derived from LMI (21). This comes from substituting (37) into (21) and applying $\lambda = 1$ based on (38). The third term of (21) can be multiplied by $\gamma^{-2}$ without loss of generality. The proof is based on the dissipation inequality satisfied by the storage function $V : \mathbb{R}^{n_{\text{vd}} \times n_{\text{vd}}} \rightarrow \mathbb{R}^+$ as $V(x_{d}) := x_d^T \bar{P} x_d$. LMI (40) can be multiplied on the left/right by $[x_d^T \bar{u}^T \ \bar{d}^T]$ and $[x_{d}\bar{u}^T \ \bar{d}^T]^T$ to show that $V$ satisfies the dissipation inequality:

$$\dot{V} = -\|u^T \bar{d}^T \| \bar{d} + \frac{1}{\sqrt{\gamma}} \|u^T \bar{d}^T \| \bar{d} = 0$$

(41)

Integrating the dissipation inequality and applying condition (35) and (38) results in $\|\bar{d}\| \leq \gamma \|d\|$ for $F_u(F_L(G_\rho, K_\rho), \bar{\Delta})$ in case $\gamma \leq 1$. The condition $\gamma \leq 1$ can be always achieved by scaling signal $\bar{d}$. $\|\bar{d}\| \leq \gamma \|d\|$ implies $\|\bar{d}\| \leq \gamma \|d\|$ based on

$$\sup_{d \neq 0, d \in \mathbb{R}^2, \rho \in \mathcal{P}, x(0)=0} \|e\| \leq \gamma \sup_{d \neq 0, d \in \mathbb{R}^2, \rho \in \mathcal{P}, x(0)=0} \|d\|$$

(42)

The inequality follows because $\sup_{d \neq 0, d \in \mathbb{R}^2, \rho \in \mathcal{P}, x(0)=0} \|e\| \leq \gamma \sup_{d \neq 0, d \in \mathbb{R}^2, \rho \in \mathcal{P}, x(0)=0} \|d\|$. Can only decrease if the constraint $\bar{d} = 0$ is added. $G_\rho$ and $\bar{\Delta}$ capture all the terms that are neglected in the Jacobian linearization of the nonlinear system $G$. Therefore, $G_\rho$ and $\bar{\Delta}$ are connected via the nonlinear system $G$.

The form of dissipation inequality (41) implies a connection to nominal induced $L_2$ gain performance. Therefore, controller $K_\rho$ can be designed based on [8], [7]. The optimal value of $\lambda$ can be found by constructing $G_\rho$ and $\bar{\Delta}$ and evaluating LMI (40) over a gridded domain of $\lambda$. The conditions of Theorem 3 can be relaxed by applying Zames-Falb multipliers and/or using multiple IQCs for $\bar{\Delta}$ and solving the synthesis problem as presented in [19].

V. EXAMPLE

A simple numerical example is presented to show the benefits of the proposed control design method. Consider the nonlinear system (similar to the example in [13]) given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} [x_1 \ x_2] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \frac{2}{1 + e^{-2x_2}} + 1, \ y = x_2$$

(43)

The aim is to design an output-feedback controller $K_\rho$, which ensures step response settling time of less than 2 seconds with zero steady state error. The scheduling parameter $\rho : x_2$ is restricted to the interval $[-10, 10]$ with a grid of 5 equidistant points. The trim points $(\bar{x}_1(\rho), \bar{u}(\rho))$ are

$$\bar{x}_1(\rho) = \bar{u}(\rho), \ \bar{x}_1(\rho) = 1 - \frac{2}{1 + e^{-2\rho}}$$

(44)

The LPV system $G_\rho$ is obtained by Jacobian linearization of (43) about the trim points. It is assumed that $\dot{\rho}$ is measurable and can be incorporated as an input to the controller in the LPV design. Four control design cases are examined as given in Table I. The linearization error terms that are accounted for in the control design are denoted by $\check{\rho}$.

The synthesis interconnection is shown in Fig. 3. Fig. 3.a gives the interconnection for Cases 1–2 (term $\Delta$ is omitted in Case 1). Fig. 3.b depicts the interconnection for Cases 3–4. The

<table>
<thead>
<tr>
<th>CONTROL DESIGN CASES</th>
<th>Taylor series error</th>
<th>Time variation of $\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>−</td>
<td>−</td>
</tr>
<tr>
<td>Case 2</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Case 3</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Case 4</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

tracking error $e_1$ is specified by weighting function $W_1$ and the control signal is penalized by the weighting function $W_2$, both with a bandwidth of $25 rad/s$ as

$$W_1(s) = \frac{0.33s + 23.69}{s + 2.369}, \ W_2(s) = \frac{0.0004s + 8.66^{-5}}{s + 43.3}$$

(45)

A robust LPV controller is designed for each case using the proposed method given in Section IV. All controllers achieve similar worst case gain of $\gamma \approx 0.12$. The effect of parameter $\lambda$ on the worst case gain $\gamma$ is shown in Fig. 4.a.
The responses of the four controllers interconnected with the nonlinear system are depicted in Fig. 4.b, which shows that Cases 2–4 clearly outperform the nominal control design.

![Fig. 4. Results of parameter Cases 2–4 with nonlinear system]

The higher order terms of the Taylor series expansion are treated as a memoryless uncertainty whose input/output behavior is described by a parameter varying IQC. The effect of the time variation of the scheduling parameter is captured by an additional disturbance. The resulting control synthesis guarantees for the interconnection of the nonlinear system and the resulting LPV controller. The benefits of the proposed method are shown by a simple numerical example. Future work will consider extending the results, with some restrictions, to non-Lipschitz continuous nonlinear systems.

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