Robust Synthesis for Linear Parameter Varying Systems Using Integral Quadratic Constraints

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Abstract

A robust synthesis algorithm is developed for a class of uncertain, linear parameter varying (LPV) systems. The uncertain system is described as an interconnection of a nominal (not-uncertain) LPV system and a block structured uncertainty. The uncertain system is described as a “gridded” LPV system with state matrices that are arbitrary functions of the parameter. The input/output behavior of the uncertainty is described by integral quadratic constraints (IQCs). The term “uncertainty” is used here in a general sense, i.e. IQCs can be used to describe not only uncertain elements but also (known) nonlinear elements, e.g. saturation. The robust synthesis problem leads to a non-convex optimization. The proposed algorithm is a coordinate-wise descent similar to the well-known DK-iteration for $\mu$ synthesis. Specifically, the proposed algorithm alternates between an LPV synthesis step and an IQC analysis step. Both steps can be efficiently solved as semidefinite programs (SDPs). There are two main technical challenges. First, the nominal LPV system does not have a valid frequency response interpretation and hence the analysis requires a time domain, dissipation inequality approach. This requires frequency domain IQCs to be converted into equivalent time domain IQCs. Second, an appropriate scaled system must be constructed to link the analysis and synthesis steps. Several technical results are provided to address these challenges. Finally, it is shown that the proposed algorithm ensures that the robust performance metric is non-increasing at each iteration step. The effectiveness of the proposed method is demonstrated on a simple numerical example.

1 INTRODUCTION

This paper considers the robust synthesis problem for a class of uncertain linear parameter varying (LPV) systems. The uncertain system is described as an interconnection of a nominal (not-uncertain) LPV system and a block structured uncertainty. The state matrices of the nominal system are assumed to have an arbitrary dependence on parameters, i.e. the nominal part is a “gridded” LPV system. Such models arise naturally in many applications via linearization of a nonlinear model around parameterized operating (trim) points. Specific examples of current interest include aeroelastic vehicles [15] and wind turbines [4,27]. The input/output behavior of the uncertainty is described by integral quadratic constraints (IQCs) [13]. The use of IQCs is sufficiently general to describe “uncertain” components that include nonlinearities, e.g. saturation, in addition to (parametric or dynamic) uncertainty.

The robust synthesis problem, formulated in Section 3.1, involves a search for a controller that minimizes a closed-loop robust performance metric. This leads to a non-convex optimization that involves a search for both the controller state matrices and the IQC analysis variables. The proposed algorithm, given in Section 3.3, consists of a coordinate-wise descent similar to the well-known DK-iteration [34,2] for $\mu$ synthesis. Specifically, the proposed algorithm alternates between an LPV synthesis step and an IQC analysis step. The synthesis step essentially relies on existing results for nominal (not uncertain) ”gridded” LPV systems in [29,30]. The analysis step is performed using a matrix inequality condition to bound the robust performance of the closed-loop uncertain LPV system (Section 4.1). Both steps can be efficiently solved as semidefinite programs (SDPs). The effectiveness of the proposed method is demonstrated on a simple numerical example in Section 5.

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There are two main technical challenges in developing this algorithm. First, the nominal LPV system does not have a valid frequency response interpretation and hence the robustness analysis requires a time domain approach. Section 4.1 develops a matrix inequality robustness analysis condition (Theorem 3) using (time domain) dissipation inequality techniques. This result requires several technical lemmas to convert a conic combination of many frequency domain IQCs into a single, equivalent time domain IQC. This analysis condition is an extension of the worst case gain condition in [16,17]. The second technical challenge is that an appropriate scaled system must be constructed to link the analysis and synthesis steps. In particular, the single equivalent time domain IQC from the analysis step must be combined with the nominal open-loop system to create the scaled system. This construction, described in Section 4.2, is such that the next synthesis step on the scaled plant yields a controller that improves the closed-loop robust performance. These technical results are used to show the following main result in Section 4.3: the robust performance metric is non-increasing at each iteration step and hence the algorithm converges.

This paper builds on many known results for both LPV systems and IQCs. A brief review of these existing results is provided in Section 2. In addition, there are several related robust synthesis results for LPV systems [1,24,25,20,19]. These existing robust synthesis results are for the case where the state matrices of the nominal LPV system have a rational dependence on the scheduling parameters. This rational (linear fractional) dependence on the parameters is exploited in the algorithm development and leads to finite-dimensional matrix inequalities for both the synthesis and analysis steps. In contrast, the algorithm in this paper is developed for the case where the state matrices of the nominal LPV system have an arbitrary dependence on the parameters. As noted above, this enables applications to systems, e.g. aeroelastic aircraft or wind turbines, for which arbitrary dependence on scheduling parameters is a natural modeling framework. The drawback of this approach is that it leads to parameter-dependent matrix inequalities for both the synthesis and analysis steps. As a result, parameter gridding is required to obtained finite-dimensional matrix inequality conditions. Finally, this paper builds on a related conference paper submission [26]. The conference paper only considered LTI uncertainty while this paper considers (possibly nonlinear) components whose input/output behavior are described by a general class of dynamic IQCs.

2 BACKGROUND

2.1 Notation

Most notation is from [34]. \( \mathbb{R}_\infty \) denotes the set of rational functions with real coefficients that are proper and have no poles on the imaginary axis. \( \mathbb{H}_\infty \) is the subset of functions in \( \mathbb{R}_\infty \) that are analytic in the closed right half of the complex plane. Vector and matrix dimensions are denoted by superscripts, e.g. \( \mathbb{R}^{m \times n}_\infty \) denotes the set of \( m \times n \) matrices whose elements are in \( \mathbb{R}_\infty \). For \( u \in L_2[0,\infty) \), \( u_T \) is the truncated function: \( u_T(t) = u(t) \) for \( t \leq T \) and \( u_T(t) = 0 \) otherwise. The extended space, denoted \( L_{2e} \), is the set of functions \( v \) such that \( v_T \in L_2 \) for all \( T \geq 0 \). The para-Hermitian conjugate of \( G \in \mathbb{R}^{m \times n}_\infty \), denoted as \( G^\sim \), is defined by \( G^\sim(s) := G(-s)^T \). Note that on the imaginary axis, \( G^\sim(j\omega) = G(j\omega)^* \).

2.2 Linear Parameter Varying Systems

Linear parameter varying (LPV) systems are a class of systems whose state-space matrices depend on a time-varying parameter vector \( \rho : \mathbb{R}_+ \to \mathbb{R}^{n_\rho} \). The parameter vector is assumed to be a continuously differentiable function of time. In addition, admissible trajectories are restricted, based on physical considerations, to lie in a known compact subset \( \mathcal{P} \subset \mathbb{R}^{n_x} \) at each point in time. In many cases, the bounds on the parameters take the simple form of a hyperrectangle, i.e. \( \mathcal{P} := \{ \rho \in \mathbb{R}^{n_\rho} | \rho_i \leq \rho_i \leq \rho_i, i = 1, \ldots, n_\rho \} \). The set of admissible trajectories is defined as \( \mathcal{T} := \{ \rho : \mathbb{R}_+ \to \mathbb{R}^{n_\rho} : \rho(t) \in \mathcal{P} \forall t \geq 0 \) and \( \rho(t) \) is continuously differentiable\}. In some applications, the parameter rates of variation \( \dot{\rho} \) are assumed to be bounded. However, only the rate-unbounded case is considered here for simplicity. All results in this paper generalize, but with more extensive notation, to the rate bounded case using existing results in [29,30,16,17].

The state-space matrices of an LPV system are continuous functions of the parameters: \( A : \mathcal{P} \to \mathbb{R}^{n_x \times n_x} \), \( B : \mathcal{P} \to \mathbb{R}^{n_x \times n_u} \), \( C : \mathcal{P} \to \mathbb{R}^{n_z \times n_x} \) and \( D : \mathcal{P} \to \mathbb{R}^{n_z \times n_u} \). An \( n_G^{th} \) order LPV system, \( G_\rho \), is defined by

\[
\begin{bmatrix}
\dot{x}(t) \\
e(t)
\end{bmatrix} = \begin{bmatrix}
A(\rho(t)) & B(\rho(t)) \\
C(\rho(t)) & D(\rho(t))
\end{bmatrix} \begin{bmatrix}
x(t) \\
d(t)
\end{bmatrix}
\]

(1)
The state matrices at time $t$ depend on the parameter vector at time $t$. Hence, LPV systems represent a special class of time-varying systems. Throughout the remainder of the paper the explicit dependence on $t$ is occasionally suppressed to shorten the notation. Moreover, it is important to emphasize that the state matrices are allowed to have an arbitrary dependence on the parameters. This is in contrast to the work in [11,20,19] where the state matrices are assumed to be rational functions of $\rho$. The performance of an LPV system $G_\rho$ can be specified in terms of its induced $L_2$ gain from input $d$ to output $e$ assuming the initial condition $x(0) = 0$, i.e. it is defined as

$$\|G_\rho\| := \sup_{\|d\|_2 \leq 1} \frac{\|e\|}{\|d\|_2}.$$  

(2)

In words, this is the largest input/output gain over all possible inputs $d \in L_2$ and allowable trajectories $\rho \in \mathcal{T}$. The notation $\rho \in \mathcal{T}$ refers to the entire (admissible) trajectory as a function of time. The analysis and synthesis theorems summarized below involve conditions on the parameters at a single point in time, i.e. $\rho(t)$. The parametric description $\rho \in \mathcal{P}$ is introduced to emphasize that such conditions only depend on the (finite-dimensional) set $\mathcal{P}$. A generalization of the Bounded Real Lemma is stated in [30] which provides a sufficient condition to upper bound the description $\rho$.

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The next theorem states the condition provided in [29,30] but simplified for the special case of rate unbounded LPV systems.

**Theorem 1.** ([29,30]): Let $\mathcal{P}$ be a given compact set and $G_\rho$ an LPV system (Equation 1). $G_\rho$ is exponentially stable and $\|G_\rho\| \leq \gamma$ if there exists a matrix $P = P^T \geq 0$ such that $\forall \rho \in \mathcal{P}$

$$
\begin{bmatrix}
PA(\rho) + A(\rho)^T P P B(\rho) \\
B^T(\rho) P
\end{bmatrix} + \frac{1}{\gamma^2} 
\begin{bmatrix}
C(\rho)^T \\
D(\rho)^T
\end{bmatrix} 
\begin{bmatrix}
C(\rho) \\
D(\rho)
\end{bmatrix} < 0
$$

(3)

**Proof.** The proof is based on a dissipation inequality satisfied by the storage function $V(x) = x^TPx$. The proof is sketched as similar arguments are used throughout the paper. Let $d \in L_2$ be an arbitrary input and $\rho \in \mathcal{T}$ be any admissible parameter trajectory. Let $x$ and $e$ denote the state and output responses of $G_\rho$ for the input $d$ and trajectory $\rho$ assuming $x(0) = 0$. Multiplying Equation 3 on the left/right by $[x^T, d^T]$ and $[x^T, d^T]^T$ gives

$$
\dot{V}(t) \leq d(t)^T d(t) - \gamma^{-2} e(t)^T e(t)
$$

(4)

Integrating this dissipation inequality yields the conclusion $\|G_\rho\| \leq \gamma$. The proof of exponential stability is similar.

This analysis theorem forms the basis for the induced $L_2$ norm controller synthesis in [29,30]. The results in [29,30] are briefly summarized for the rate unbounded case. Consider an open loop LPV system $G_\rho$ defined as

$$
\begin{bmatrix}
\dot{x} \\
e \\
y
\end{bmatrix} =
\begin{bmatrix}
A(\rho) & B_d(\rho) & B_u(\rho) \\
C_e(\rho) & D_{ed}(\rho) & D_{eu}(\rho) \\
C_y(\rho) & D_{yd}(\rho) & D_{yu}(\rho)
\end{bmatrix}
\begin{bmatrix}
x \\
d \\
u
\end{bmatrix}
$$

(5)

where $x \in \mathbb{R}^{n_x}$, $d \in \mathbb{R}^{n_d}$, $e \in \mathbb{R}^{n_e}$, $u \in \mathbb{R}^{n_u}$ and $y \in \mathbb{R}^{n_y}$. The goal is to synthesize an LPV controller $K_\rho$ of the form:

$$
\begin{bmatrix}
\dot{x}_K \\
u
\end{bmatrix} =
\begin{bmatrix}
A_K(\rho) & B_K(\rho) \\
C_K(\rho) & D_K(\rho)
\end{bmatrix}
\begin{bmatrix}
x_K \\
y
\end{bmatrix}
$$

(6)

The controller generates the control input $u$. It has a linear dependence on the measurement $y$ but an arbitrary dependence on the (measurable) parameter vector $\rho$. The closed-loop interconnection of $G_\rho$ and $K_\rho$ is given by a lower linear fractional transformation (LFT) and is denoted $\mathcal{F}_\Gamma(G_\rho, K_\rho)$. The objective is to synthesize a controller.
Let $K_\rho$ of the specified form to minimize the closed-loop induced $L_2$ gain from disturbances $d$ to errors $e$:

$$
\min_{K_\rho} \| \mathcal{F}_l(G_\rho, K_\rho) \|.
$$

(7)

The notation for the synthesis result below is greatly simplified by assuming the feedthrough matrices satisfy $D_{\text{out}}(\rho) = 0$, $D_{\text{in}}(\rho) = 0$ and $D_{\text{out}}(\rho)^T = [0, I_{n_x}]$, $D_{\text{in}}(\rho) = [0, I_{n_y}]$. Under some technical rank assumptions, this normalized form can be achieved through a combination of loop-shifting and scaling [29,18]. The input matrix is partitioned as $B_d := [B_{d1}(\rho) \, B_{d2}(\rho)]$ compatibly with the normalized form of $D_{\text{in}}$. Similarly, the output matrix is partitioned as $C_d^T := [C_{d1}^T(\rho) \, C_{d2}^T(\rho)]$ compatibly with $D_{\text{out}}$. Given these simplifying assumptions, the solution to the induced $L_2$ control synthesis problem is stated in the next theorem.

**Theorem 2** ([29,30]). Let $\mathcal{P}$ be a given compact set and $G_\rho$ an LPV system (Equation 5) that satisfies the normalizing assumptions above. There exists a controller $K_\rho$ as in Equation 6 such that $\| \mathcal{F}_l(G_\rho, K_\rho) \| \leq \gamma$ if there exist matrices $X = X^T > 0$ and $Y = Y^T > 0$ such that $\forall \rho \in \mathcal{P}$

$$
\begin{bmatrix}
X & I_{n_x} \\
I_{n_y} & Y
\end{bmatrix} \succeq 0
$$

(8)

$$
\begin{bmatrix}
Y \dot{A}(\rho)^T + \dot{A}(\rho) Y - \gamma B_w(\rho) B_w(\rho)^T Y C_{e1}(\rho)^T B_d(\rho) \\
C_{e1}(\rho)^T Y & -\gamma I_{n_x1} \\
B_d(\rho)^T & 0 & -\gamma I_{n_x2}
\end{bmatrix} < 0
$$

(9)

$$
\begin{bmatrix}
\dot{A}(\rho)^T X + X \dot{A}(\rho) - C_y(\rho)^T C_y(\rho) X B_{d1}(\rho) C_e(\rho)^T \\
B_{d1}(\rho)^T X & -\gamma I_{n_x1} \\
C_e(\rho) & 0 & -\gamma I_{n_x2}
\end{bmatrix} < 0
$$

(10)

where $\dot{A}(\rho) := A(\rho) - B_w(\rho) C_{e2}(\rho)$ and $\dot{A}(\rho) := A(\rho) - B_{d2}(\rho) C_y(\rho)$.

**Proof.** The proof uses a matrix elimination argument similar to that used in the LMI approach to $H_\infty$ synthesis for linear time invariant (LTI) systems [10].

If the conditions in Theorem 2 are satisfied then an LPV controller $(A_K(\rho), B_K(\rho), C_K(\rho), D_K(\rho))$ can be constructed from the open loop plant matrices and the feasible values of $X$, $Y$, and $\gamma$. The controller reconstruction procedure is given in [29,30]. Moreover, a storage function matrix $P \succeq 0$ can be constructed from $X$ and $Y$ such that the closed-loop satisfies the nominal performance LMI condition (Equation 3) in Theorem 1. Finally, the closed-loop performance (upper bound) can be optimized by minimizing $\gamma$ subject to the LMI constraints in Theorem 2. This yields a semidefinite programming formulation for the LPV synthesis problem.

### 2.3 Integral Quadratic Constraints

Integral quadratic constraints (IQCs) [13] provide a framework for robustness analysis building on work by Yakubovich [31]. The IQC specifies a constraint on the input-output signals of the perturbation. The form of the constraint is such that it can be easily incorporated into tractable stability and performance analysis conditions. The following definitions characterize the constraint in the frequency and time domain.

**Definition 1.** Let $\Pi$ be a rational and uniformly bounded function of $j\omega$, i.e.

$$
\Pi \in \mathbb{R}^{(m_1+m_2) \times (m_1+m_2)}
$$

Two signals $v \in L_2^{n_\nu}[0, \infty)$ and $w \in L_2^{n_w}[0, \infty)$ satisfy the frequency domain IQC defined by the multiplier $\Pi$ if

$$
\int_{-\infty}^{\infty} \begin{bmatrix}
\dot{V}(j\omega) \\
W(j\omega)
\end{bmatrix}^* \Pi(j\omega) 
\begin{bmatrix}
\dot{V}(j\omega) \\
W(j\omega)
\end{bmatrix} \, d\omega \geq 0
$$

(11)
where \( \hat{V} \) and \( \hat{W} \) are Fourier transforms of \( v \) and \( w \). A bounded, causal operator \( \Delta : L_{2e}^n[0, \infty) \to L_{2e}^w[0, \infty) \) satisfies the frequency domain IQC defined by \( \Pi \) if Equation 11 holds for all \( v \in L_{2e}^n[0, \infty) \) and \( w = \Delta(v) \).

**Definition 2.** Let \( \Psi \) be a stable LTI system, i.e., \( \Psi \in \mathbb{R}^{n_x \times n_y} \), and \( M = M^T \in \mathbb{R}^{n_z \times n_y} \). Two signals \( v \in L_2^n[0, \infty) \) and \( w \in L_2^w[0, \infty) \) satisfy the time domain IQC defined by the multiplier \( \Psi \) and matrix \( M \) if the following inequality holds for all \( T \geq 0 \)

\[
\int_0^T z^T(t)MZ(t)dt \geq 0
\]  

(12)

where \( z \) is the output of \( \Psi \) driven by inputs \( (v, w) \) with zero initial conditions. A bounded, causal operator \( \Delta : L_{2e}^n[0, \infty) \to L_{2e}^w[0, \infty) \) satisfies the time domain IQC defined by \( (\Psi, M) \) if Inequality 12 holds for all \( v \in L_{2e}^n[0, \infty) \), \( w = \Delta(v) \) and \( T \geq 0 \).

IQC's can be used to model a variety of nonlinearities and uncertainties. In particular, [13] provides a library of frequency domain IQC multipliers that are satisfied by many important system components, e.g., saturation, time delay, and norm bounded uncertainty. Figure 1 provides a graphical interpretation for the time domain IQC. The input and output signals of \( \Delta \) are filtered through \( \Psi \). If \( \Delta \) satisfies the time domain IQC defined by \( \Psi \) then the filtered signal \( z \) satisfies the constraint in Equation 12 for any finite-horizon \( T \geq 0 \).

![Fig. 1. Graphical interpretation of the IQC](image)

A precise connection between the frequency and time domain IQC formulations is important for the robust synthesis algorithm described in this paper. Assume \( \Delta \) satisfies the time domain IQC defined by \( (\Psi, M) \). Taking \( T \to \infty \) in Equation 12 yields \( \int_0^\infty z(t)^TMz(t)dt \geq 0 \). By Parseval's theorem [34], this is equivalent to the frequency domain constraint \( \int_0^\infty \hat{Z}(j\omega)^*\hat{M}\hat{Z}(j\omega)d\omega \geq 0 \) where \( \hat{Z}(j\omega) = \Psi(j\omega)\frac{\hat{V}(j\omega)}{\hat{W}(j\omega)} \). Thus if \( \Delta \) satisfies the time domain IQC defined by \( (\Psi, M) \) then it satisfies the frequency domain IQC defined by \( \Pi = \Psi^M\Psi \).

The reverse implication is more technical and fails to hold in general. Specifically, assume \( \Delta \) satisfies the frequency domain IQC defined by the multiplier \( \Pi \). Any rational multiplier \( \Pi \) can be factorized as \( \Pi = \Psi^M\Psi \) where \( \Psi \in \mathbb{R}^{n_x \times n_y} \) is stable and \( M = M^T \in \mathbb{R}^{n_z \times n_y} \). Such factorizations are not unique but can be computed using state-space calculations [21,9,5]. One specific numerical construction is given by Lemma 4 in Appendix A. Substitute the factorization for \( \Pi \) into the frequency domain IQC (Equation 11) and apply Parseval's theorem [34] to convert to a time domain constraint. This yields \( \int_0^\infty z(t)^TMz(t)dt \geq 0 \) where \( z \) is the output of \( \Psi \) driven by \( v \) and \( w = \Delta(v) \) with zero initial conditions. This time domain constraint holds, in general, only over infinite horizons and only for finite-norm input signals \( v \in L_2^n[0, \infty) \). However, the time domain IQC (Definition 2) requires the integral inequality to hold over all finite times \( T \geq 0 \) and for all extended-space input signals \( v \in L_{2e}^n[0, \infty) \). A time domain IQC as in Definition 2 is referred to as a hard IQC in [13]. In contrast, factorizations for which the time domain constraint holds only for \( T = \infty \) are called soft IQCs. This distinction is important because the dissipation theorems specified later for robustness analysis require the use of hard IQCs. Lemmas 5 and 6 in Appendix A provide a specific "hard" factorization \((\Psi, M)\) that can be constructed under additional assumptions on the frequency domain multiplier \( \Pi \).

To summarize these lemmas, let \( \Pi = \Pi^\sim \in \mathbb{R}^{n_x \times n_x} \times (n_x + n_u) \) be partitioned as \( [\Pi_{11} \Pi_{12} \Pi_{21} \Pi_{22}] \) where \( \Pi_{11} \in \mathbb{R}^{n_x \times n_x} \) and \( \Pi_{22} \in \mathbb{R}^{n_x \times n_u} \). If \( \Pi_{11}(j\omega) > 0 \) and \( \Pi_{22}(j\omega) < 0 \) \( \forall \omega \in \mathbb{R} \cup \{\infty\} \), then \( \Pi \) has a hard factorization \((\Psi, M)\) that yields a time domain IQC (Definition 2). \((\Psi, M)\) can be constructed from the stabilizing solution to an Algebraic Riccati Equation (ARE) and is called a J-spectral factorization of \( \Pi \).

\footnote{The terms "complete" and "conditional" IQCs in [12] are generalizations of hard and soft IQCs. The hard/soft terminology will be used here.}
3 Robust Synthesis

3.1 Problem Formulation

Consider the robust synthesis problem for the uncertain LPV system as shown in Figure 2. The uncertain LPV system is described by the interconnection of an open loop LPV system $G_\rho$, a perturbation $\Delta$, and an LPV controller $K_\rho$. A state-space realization for $G_\rho$ is given by:

$$
\begin{align*}
\dot{x}_G &= \begin{bmatrix} A(\rho) & \mathcal{B}_w(\rho) & \mathcal{B}_d(\rho) & \mathcal{B}_u(\rho) \\ \mathcal{C}_v(\rho) & \mathcal{D}_{cv}(\rho) & \mathcal{D}_{cd}(\rho) & \mathcal{D}_{cu}(\rho) \\ \mathcal{C}_e(\rho) & \mathcal{D}_{ce}(\rho) & \mathcal{D}_{cd}(\rho) & \mathcal{D}_{ce}(\rho) \\ \mathcal{C}_y(\rho) & \mathcal{D}_{cy}(\rho) & \mathcal{D}_{dy}(\rho) & \mathcal{D}_{ey}(\rho) \end{bmatrix} \begin{bmatrix} x_G \\ v \\ c \\ y \end{bmatrix},
\end{align*}
$$

where $x_G \in R^{n_G}$, $w \in R^{n_w}$, $d \in R^{n_d}$, $u \in R^{n_u}$, $v \in R^{n_v}$, $c \in R^{n_c}$ and $y \in R^{n_y}$. The following assumptions are made regarding $G_\rho$ and $\Delta$:

**Assumption 1.** $G_\rho$ is quadratically stabilizable from $u$ and quadratically detectable from $y$ as defined in Chapter 1 of [29].

**Assumption 2.** The perturbation is a bounded, causal operator $\Delta : L_0^{n_u}[0, \infty) \rightarrow L_\infty^{n_u}[0, \infty)$ that satisfies a collection of frequency domain IQCs defined by $\{\Pi_k\}_{k=1}^N \subset \mathbb{R}^{(n_u+n_v) \times (n_u+n_v)}$.

**Assumption 3.** Partition the frequency domain multipliers $\{\Pi_k\}_{k=1}^N$ as $[\Pi_{k,11} \Pi_{k,12} \mid \Pi_{k,21} \Pi_{k,22}]$ where $\Pi_{k,11}$ is $n_v \times n_v$. Each frequency domain multiplier satisfies $\Pi_{k,11}(j\omega) \geq 0$ and $\Pi_{k,22}(j\omega) \leq 0 \ \forall\omega \in \mathbb{R} \cup \{\infty\}$.

**Assumption 4.** The perturbation has been normalized to satisfy $\|\Delta\| \leq 1$ and the first IQC is defined by the multiplier $\Pi_1 := \begin{bmatrix} I_{n_u} & 0 \\ 0 & -I_{n_y} \end{bmatrix}$.

![Fig. 2. Interconnection for LPV Robust Synthesis](image)

The first assumption ensures that there is a controller $K_\rho$ from $y$ to $u$ that stabilizes the (nominal) open loop interconnection of $G_\rho$ and $K_\rho$. This open loop interconnected system is a lower LFT, denoted $F_l(G_\rho, K_\rho)$. The IQCs in the second assumption are used to bound the input-output behavior of the perturbation $\Delta$. This formulation can handle systems where $\Delta$ has block diagonal structure including static nonlinearities (e.g. saturations) and infinite dimensional operators (e.g. time delays) in addition to true system uncertainties. The term "uncertainty" is used for simplicity when referring to the perturbation $\Delta$. The notation $\Delta(\Pi_1, \ldots, \Pi_N)$ will be used to denote the set of bounded, causal operators $\Delta$ that satisfy all frequency domain IQCs defined by $\{\Pi_k\}_{k=1}^N$.

The third and fourth assumptions are used to simplify the algorithm and can be relaxed. Assumption 3 only requires the multipliers satisfy non-strict definiteness conditions $\Pi_{k,11} \geq 0$ and $\Pi_{k,22} \leq 0$. This is sufficiently general to cover most typical frequency domain multipliers used in IQC analysis. In fact, all frequency domain multipliers listed in [13] satisfy $\Pi_{k,11} \geq 0$ and $\Pi_{k,22} \leq 0$ except those for certain sector bounded nonlinearities and polytopic uncertainties which fail to contain the zero operator $\Delta = 0$. Finally, note that the individual multiplier $\Pi_k$ need not satisfy the strict definiteness conditions $\Pi_{k,11} > 0$ and $\Pi_{k,22} < 0$ given in Lemma 5 for the existence of a $J$-spectral factorization. However, Assumptions 3 and 4 are sufficient to ensure that a "combined" multiplier that appears in the proposed robust synthesis algorithm satisfies the strict definiteness conditions and thus has a $J$-spectral factorization.

To simplify notation, define $H_\rho := F_l(G_\rho, K_\rho)$. The uncertain LPV system in Figure 2 can therefore be expressed as
an upper LFT, denoted $\mathcal{F}_u(H_\rho, \Delta)$. A natural performance metric for the uncertain LPV system is the worst-case gain:

$$\sup_{\Delta \in \Delta_{\Pi_1, \ldots, \Pi_N}} \| \mathcal{F}_u(H_\rho, \Delta) \|$$ (14)

This is the largest induced $L_2$ gain of the uncertain LPV system over all uncertainties consistent with the specified IQCs. This metric is inconvenient for robust synthesis as it requires an initial controller that achieves robust stability with respect to $\Delta(\Pi_1, \ldots, \Pi_N)$. Thus it is standard, e.g. in DK synthesis, to instead use a robust performance metric that simultaneously scales both the uncertainty level and the system gain. This metric, formally defined below, is used for the robust synthesis algorithm in this paper. The definition of robust performance requires the notion of a scaled uncertainty set. Specifically, define $\mathcal{S}_b$ as the scaling matrix $\begin{bmatrix} bI_n & 0 \\ 0 & I_{n_w} \end{bmatrix}$. Let $\Delta_{b}(\Pi_1, \ldots, \Pi_N)$ denote the set of bounded, causal operators $\Delta$ that satisfy the frequency domain IQCs defined by $\mathcal{S}_b \Pi_k \mathcal{S}_b^T$ for $k = 1, \ldots, N$. Note that $\Delta \in \Delta_b$ if and only if $\Delta \circ \frac{1}{b} \in \Delta$. Hence $b > 1$ typically implies the scaled set $\Delta_b$ is larger than the unscaled set $\Delta$.

**Definition 3.** The system $H_\rho$ achieves robust performance of level $\gamma$ with respect to the uncertainty described by $\{\Pi_k\}_{k=1}^N$ if

$$\sup_{\Delta \in \Delta_{1/\gamma}(\Pi_1, \ldots, \Pi_N)} \| \mathcal{F}_u(H_\rho, \Delta) \| \leq \gamma$$ (15)

Let $r_{\Delta(\Pi_1, \ldots, \Pi_N)}[H_\rho]$ denote the smallest level of robust performance achievable by $H_\rho$.

$H_\rho$ achieves robust performance of level $\gamma$ if the worst-case induced $L_2$ gain from $d$ to $e$ is $\leq \gamma$ over all uncertainties in the scaled set $\Delta_{1/\gamma}(\Pi_1, \ldots, \Pi_N)$. For decreasing levels of robust performance, the gain decreases and the bound of the tolerable uncertainty increases. The robust performance level for the uncertain system with known nominal plant $H_\rho$ can be analyzed using convex optimization as described in Section 4.1.

The objective of the robust synthesis problem is to synthesize an LPV controller $K_\rho$ with the form in Equation 6 that stabilizes the open-loop model $G_\rho$ and minimizes the closed-loop robust performance. Thus the synthesis problem is:

$$\inf_{K_\rho \text{stabilizing}} r_{\Delta(\Pi_1, \ldots, \Pi_N)}[\mathcal{F}_l(G_\rho, K_\rho)]$$ (16)

### 3.2 DK Synthesis

This section briefly reviews the standard DK synthesis algorithm [34,2]. The objective is to clarify the notation presented thus far and to provide a basis for comparison with the proposed algorithm. In DK synthesis the nominal plant $G$ is LTI and the uncertainty $\Delta$ is LTI and unit norm bounded. The robust synthesis problem involves the search for an LTI controller $K$ and robustness analysis scaling $D$ (called $D$-scales). The problem is non-convex, in general, and DK synthesis employs a coordinate-wise iteration. Specifically, the algorithm iterates between a controller synthesis step ($K$-step) and a robustness analysis step ($D$-step). The synthesis step involves the design of an $H_\infty$ controller $K$ on a nominal (not-uncertain) scaled system. The analysis step involves the search for a frequency domain scaling $D$ to assess the robust performance of the closed-loop $H := F_l(G, K)$. The coordinate-wise iteration for DK synthesis does not, in general, converge to a local (nor global) optima. However it has the advantage that each of the decoupled synthesis and analysis steps is a convex optimization.

The main technical result for DK synthesis is that the iteration is well posed at each step and the robust performance is (in theory) non-increasing. This result is based on the construction of a scaled system that links the analysis and synthesis steps. The scaled system used in the $K$-step is $DGD^{-1}$ where $D$ is the scaling from the analysis step. The main loop theorem [34] establishes the equivalence between robust performance of the (uncertain) closed-loop $\mathcal{F}_l(H, \Delta)$ and the induced $L_2$ performance of the (not-uncertain) scaled system $\mathcal{F}_l(DGD^{-1}, K)$. If the synthesis problem includes mixed (real and complex) uncertainty then the construction of an appropriate scaled system is more subtle. For example, the DGK synthesis algorithm [32,33] uses a specific factorization of the $D/G$ scalings to
prove that the robust performance monotonically decreases. One of the major technical results given below leads to the construction of an appropriate scaled system for the robust LPV synthesis with general IQCs.

Finally, we briefly connect the notation used in standard DK synthesis with that introduced here for the robust LPV synthesis problem. The uncertainty in DK synthesis is, in general, block structured but for simplicity this discussion assumes Δ is SISO (no structure). If ∥Δ∥∞ ≤ 1 then Δ satisfies the frequency domain IQC defined by Π = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} for any SISO, LTI system \( \alpha \) such that \( \alpha(j\omega) = \alpha(j\omega)^* > 0 \) ∀\( \omega \in \mathbb{R} \cup \{\infty\} \). Moreover, if ∥Δ∥∞ ≤ \( \frac{1}{\gamma} \) then Δ satisfies the frequency domain IQC defined by the scaled multiplier \( S_{1/\gamma} \Pi S_{1/\gamma} \). The condition \( \alpha > 0 \) ensures that \( \alpha \) has a spectral factorization \( \alpha = d^*d \) where \( d \) is the scaling/multiplier that appears in DK synthesis. In this case, \( \Psi = \begin{bmatrix} d & 0 \\ 0 & 0 \end{bmatrix} \) and \( M = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \) defines a J-spectral factorization of \( \Pi \). The D-step in DK synthesis is typically implemented by solving for \( D \)-scales on a frequency grid and then fitting the result with a rational transfer function. Here, the scalings will be restricted to a finite, linear combination of user selected basis functions. In particular, the definition of robust performance (Definition 3) requires a finite number \( N \) of (fixed) multipliers \( \{\Pi_k\} \) to be specified. In the context of this DK synthesis example, this corresponds to the selection of \( N \) scalings \( \{\alpha_k\} \). The proposed algorithm given below will search for the best linear combination of these scalings.

3.3 Algorithm

This section gives a high-level overview of the proposed LPV robust synthesis algorithm. Technical details regarding the algorithm are then given in Section 4. As in DK synthesis, the robust LPV synthesis is, in general, non-convex. In particular, Theorem 3 in Section 4.1 provides a linear matrix inequality (LMI) formulation for robust performance. The detailed steps of the algorithm including the initialization and termination conditions are described in Algorithm 1. This algorithm is briefly described to provide a roadmap for the technical results in the following section. The algorithm initialization (Step 2) computes a factorization for each IQC multiplier. Any stable factorization of the \( \Pi_k \) may be used in and the construction in Lemma 4 of Appendix A is just one possibility. As noted above, a J-spectral factorization need not exist for the individual multipliers \( \Pi_k \). The main steps of the algorithm involve a synthesis step (Step 8), analysis step (Step 9), and the construction of a scaled-system \( G_{\Pi}^{\eta} \) that links these steps (Steps 5-7). The synthesis step is a standard (nominal) LPV synthesis on the scaled system. It uses the algorithm in [29,30] and summarized by Theorem 2 in Section 2.2. The analysis step involves an parameterized matrix inequality condition (Theorem 3 in Section 4.1) that involves a storage function matrix \( P \), analysis vector \( \lambda \), and robust performance bound \( \gamma \). The bound \( \gamma \) enters bilinearly in the matrix inequality and hence this step requires bisection to find the minimum feasible value of \( \gamma \). This step can be interpreted as a search over linear combinations of the scaled IQCs to form a single combined IQC multiplier (Step 6). The technical condition \( \lambda_1(i) > 0 \) in Step 9 is used to ensure that the combined IQC multiplier in Step 6 has a J-spectral factorization. Finally, the analysis and synthesis steps are linked by the construction of a particular scaled system (Step 7). The construction of the scaled system is described further in Section 4.2. The algorithm can be easily modified to incorporate other termination criteria in Step 10, e.g. maximum number of iterations and/or relative stopping tolerances.

As noted above, the LPV robust synthesis problem inherits the non-convexity of DK synthesis. The proposed coordinate-wise iteration will not, in general, converge to a local (nor global) optima. However, it is a pragmatic approach that decouples the synthesis and analysis steps into convex optimizations. The main technical result (Theorem 4 in Section 4.3) is that the algorithm iteration is well posed at each step and the robust performance is non-increasing. This is similar to the convergent property of the DK synthesis.

4 Technical Results

4.1 Robust Performance Condition

This section derives a matrix inequality condition to bound the robust performance for an uncertain LPV system. The uncertain LPV system is specified by the interconnection \( \mathcal{F}_\eta(H_\rho, \Delta) \). The main technical issue is that the uncertainty \( \Delta \) is described by IQCs \( \{\Pi\}_k=1^N \) in the frequency domain but the nominal system \( H_\rho \) is LPV and does not have a
The nominal LPV system which involves a dissipation inequality characterization for robust performance. This leads to the main technical result (Theorem 3) valid frequency domain interpretation. The approach given here combines the frequency domain IQCs and the single time domain IQC. This results in the synthesis for LPV systems. Theorem 1 gives the following state-space realization:

\[
\begin{bmatrix}
  \dot{x}_H \\
  v \\
  e
\end{bmatrix} = 
\begin{bmatrix}
  A(\rho) & B_w(\rho) & B_d(\rho) \\
  C_v(\rho) & D_{vw}(\rho) & D_{vd}(\rho) \\
  C_e(\rho) & D_{ve}(\rho) & D_{ed}(\rho)
\end{bmatrix}
\begin{bmatrix}
  x_H \\
  w \\
  d
\end{bmatrix}
\]

where \( x_H \in \mathbb{R}^{n_H}, w \in \mathbb{R}^{n_w}, d \in \mathbb{R}^{n_d}, v \in \mathbb{R}^{n_v} \) and \( e \in \mathbb{R}^{n_e} \). The uncertainty \( \Delta \) is assumed to satisfy multiple

\[
\begin{align*}
\sum \Delta_i &\leq 1 \\
\sum (\Delta_i \otimes I_{n_v}) &\leq 1 \\
\sum (\Delta_i \otimes I_{n_d}) &\leq 1
\end{align*}
\]
Equation 18 uses an abbreviated notation to denote that the outputs of $\Psi_1$ are \( \tilde{\text{realization}}: \{ \text{all } k \} \) the scaled system $\Psi_1$. This corresponds to the use of the scaled (frequency domain) multipliers $S_k, k = 1, \ldots, N$). Thus a factorization for each scaled multiplier is given by $(\Psi_k S_k, M_k)$. Let $z_k$ denote the output of the scaled system $\Psi_k S_k$ driven by the input/output signals $(v, w)$ of $\Delta$ assuming zero initial conditions. Then all \( \{ \Psi_k S_k \} \) can be aggregated into a single system denoted $\Psi_1$ with the following (minimal) state-space realization:

\[
\begin{bmatrix}
\dot{x}_v(t) \\
\dot{z}_k(t)
\end{bmatrix} =
\begin{bmatrix}
\tilde{A} & \gamma^{-1} \tilde{B}_v & \tilde{B}_w \\
\tilde{C}_v & \gamma^{-1} \tilde{D}_{vk} & \tilde{D}_{wk}
\end{bmatrix}
\begin{bmatrix}
x_v(t) \\
v(t) \\
w(t)
\end{bmatrix} \quad (k = 1, \ldots, N)
\]  

Equation 18 uses an abbreviated notation to denote that the outputs of $\Psi_1$ are $[z_1^T, \ldots, z_N^T]^T$. Note that the scaling matrix $S_1 := \begin{bmatrix} \gamma^{-1} I_{n_v} & 0 \\
0 & I_{n_w} \end{bmatrix}$ only modifies the state matrices of $\Psi_1$ associated with the $v$ input, i.e. it only scales the $\tilde{B}_v$ and $\tilde{D}_{vk}$ matrices.

The robust performance analysis is based on the interconnection shown in Figure 4 with $\Delta \in \Delta_1/\gamma(\Pi_1, \ldots, \Pi_N)$. The dynamics of this analysis interconnection are described by $w = \Delta(v)$ and the extended system of $H_\rho$ and $\Psi_1$:

\[
\begin{bmatrix}
\dot{x} \\
\dot{z}_k \\
e
\end{bmatrix} =
\begin{bmatrix}
A(\rho) & B_v(\rho) & B_d(\rho) \\
C_v(\rho) & D_{vk}(\rho) & D_{vd}(\rho) \\
C_v(\rho) & D_{cv}(\rho) & D_{vd}(\rho)
\end{bmatrix}
\begin{bmatrix}
x \\
w \\
d
\end{bmatrix} \quad (k = 1, \ldots, N)
\]  

where the state vector is $x = [x_H; x_v] \in \mathbb{R}^{n_H+n_v}$ with $x_H$ and $x_v$ being the state vectors of the LPV system $H_\rho$ and the filter $\Psi_1$, respectively. The state matrices for the extended system can be expressed in terms of the state matrices for $H_\rho$ (Equation 17) and $\Psi_1$ (Equation 18). Appendix B provides one realization. Note that the state matrices of the extended system depend on the robust performance level $\gamma$. However this dependence on $\gamma$ is not explicitly denoted. The uncertainty $\Delta$ is shown in the dashed box of Figure 4 to signify that the analysis condition given below is specified only in terms of the extended system of $H_\rho$ and $\Psi_1$. This effectively overbounds the precise relation $w = \Delta(v)$ with the IQCs satisfied by $\Delta$.

![Fig. 4. Uncertain LPV system extended to include filter $\Psi_1$](image)

The robust performance analysis condition (given below) relies on a connection between $\Psi_1$ and a combined multiplier $\Pi := \sum_{k=1}^N \lambda_k S_k/\gamma, \Pi_k S_k/\gamma$ defined for some scalars $\lambda_k \geq 0$. This combined multiplier can be expressed
in terms of the state-space realization of $\Psi_{1/\gamma}$ (Equation 18) as:

$$
\Pi_\lambda = \begin{bmatrix} (sI - \hat{A})^{-1} \hat{B} \end{bmatrix}^T \begin{bmatrix} \hat{Q}_\lambda \hat{S}_\lambda \\ \hat{S}_\lambda^T \hat{R}_\lambda \end{bmatrix} \begin{bmatrix} (sI - \hat{A})^{-1} \hat{B} \end{bmatrix}
$$

(20)

where

$$
\hat{B} := \left[ \gamma^{-1} \hat{B}_v \ \hat{B}_w \right]
$$

(21)

$$
\begin{bmatrix} \hat{Q}_\lambda \hat{S}_\lambda \\ \hat{S}_\lambda^T \hat{R}_\lambda \end{bmatrix} := \sum_{k=1}^{N} \lambda_k \begin{bmatrix} \tilde{C}_{zk}^T \\ \gamma^{-1} \tilde{D}_{zk}^w \end{bmatrix} M_k \begin{bmatrix} \tilde{C}_{zk} \gamma^{-1} \tilde{D}_{zk}^r \end{bmatrix}
$$

(22)

If $(\Pi_\lambda_{11})(j\omega) > 0$ and $(\Pi_\lambda_{22})(j\omega) < 0 \ \forall \omega \in \mathbb{R} \cup \{+\infty\}$ then $\Pi_\lambda$ has a $J$-spectral factorization (Lemma 5). This factorization is constructed from the stabilizing solution $X$ to the ARE in Equation A.2 with $(\hat{A}, \hat{B}, \hat{Q}_\lambda, \hat{S}_\lambda, \hat{R}_\lambda)$. Without loss of generality, the $J$-spectral factorization can be rescaled as $(\Psi_\lambda, M_\lambda)$ where the constant matrix is $M_\lambda := \begin{bmatrix} \gamma^{-2I} & 0 \\ 0 & -I \end{bmatrix}$. Specifically, Let $(\Psi, M)$ be a $J$-spectral factorization of $\Pi_\lambda$ with $M := \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$. Then $(\Psi_\lambda, M_\lambda) := (S_1\Psi, S_{1/\gamma}MS_{1/\gamma})$ is another factorization of $\Pi_\lambda$ with the constant matrix given by $S_{1/\gamma}$

$$
S_{1/\gamma} := \begin{bmatrix} \gamma^{-2I} & 0 \\ 0 & -I \end{bmatrix}.
$$

In addition, the properties of a $J$-spectral factorization given in Lemma 6 carry over for this rescaled factorization. This rescaling will be important for the construction of the scaled plant in the synthesis step of our proposed algorithm (described in Section 4.2). The rescaled filter $\Psi_\lambda$ only has one output and has a state-space realization of the form:

$$
\begin{bmatrix} \dot{x}_\psi(t) \\ z_\lambda(t) \end{bmatrix} = \begin{bmatrix} \hat{A} \gamma^{-1} \hat{B}_v \ \hat{B}_w \\ \hat{C}_{zk} \hat{D}_{zk}^w \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}
$$

(23)

This rescaled system $\Psi_\lambda$ has the same state matrix $\hat{A}$ and input matrix $[\gamma^{-1} \hat{B}_v, \hat{B}_w]$ as the original filter $\Psi_{1/\gamma}$. Only the output and feedthrough matrices of $\Psi_\lambda$ are different from those in $\Psi_{1/\gamma}$. Finally, an extended system of $H_\rho$ and $\Psi_\lambda$ can be formed yielding:

$$
\begin{bmatrix} \dot{x} \\ z_\lambda \\ c \end{bmatrix} = \begin{bmatrix} \mathcal{A}(\rho) & B_w(\rho) & B_d(\rho) \\ \hat{C}_{zk}(\rho) & D_{zk}^w(\rho) & D_{zk}^d(\rho) \\ \hat{C}_c(\rho) & D_{cw}(\rho) & D_{cd}(\rho) \end{bmatrix} \begin{bmatrix} x \\ v \\ d \end{bmatrix}
$$

(24)

The state matrices for this extended system can be expressed in terms of the state matrices for $H_\rho$ (Equation 17) and $\Psi_\lambda$ (Equation 23). Only the output and feedthrough matrices associated with $z_\lambda$ in this alternative extended system differ from those given in Equation 19. Appendix B provides explicit formulae for $\hat{C}_{zk}$, $D_{zk}^w$, and $D_{zk}^d$. Again, the dependence on $\gamma$ is not made explicit in this notation for the alternative extended system.

The robust performance condition relies on a technical lemma regarding matrix inequalities associated with the two extended systems presented thus far. Specifically, the extended system of $H_\rho$ and $\Psi_{1/\gamma}$ (Equation 19) can be used to define the following parameterized matrix inequality involving multiple IQCs:

$$
\begin{bmatrix} \tilde{P}_1 + \tilde{A}^T \tilde{P} \tilde{P}_B \tilde{B} & \tilde{B}_w & \tilde{B}_d \\ \tilde{B}_w^T \tilde{P} & 0 & 0 \\ \tilde{B}_d^T \tilde{P} & 0 & -I \end{bmatrix} + \sum_{k=1}^{N} \lambda_k \begin{bmatrix} \tilde{C}_{zk}^T \\ \tilde{D}_{zk}^w \\ \tilde{D}_{zk}^d \end{bmatrix} M_k \begin{bmatrix} \tilde{C}_{zk} \tilde{D}_{zk}^w \tilde{D}_{zk}^d \end{bmatrix} + \frac{1}{\gamma^2} \begin{bmatrix} \tilde{C}_c^T \\ \tilde{D}_{cw} \\ \tilde{D}_{cd} \end{bmatrix} \begin{bmatrix} \tilde{C}_c \tilde{D}_{cw} \tilde{D}_{cd} \end{bmatrix} < 0
$$

(25)

This matrix inequality is parameterized by $\rho \in \mathcal{P}$ through the dependence of the extended state system matrices on the parameter. Similarly, the extended system of $H_\rho$ and $\Psi_\lambda$ (Equation 24) can be used to define the following parameterized matrix inequality involving the single, rescaled $J$-spectral factorization:

$$
\begin{bmatrix} \tilde{P}_1 + \tilde{A}^T \tilde{P} \tilde{P}_B \tilde{B} & \tilde{B}_w & \tilde{B}_d \\ \tilde{B}_w^T \tilde{P} & 0 & 0 \\ \tilde{B}_d^T \tilde{P} & 0 & -I \end{bmatrix} + \frac{1}{\gamma^2} \begin{bmatrix} \tilde{C}_{zk}^T \\ \tilde{D}_{zk}^w \\ \tilde{D}_{zk}^d \end{bmatrix} M_k \begin{bmatrix} \tilde{C}_{zk} \tilde{D}_{zk}^w \tilde{D}_{zk}^d \end{bmatrix} < 0
$$

(26)
The technical result regarding these two matrix inequalities is formally stated in the next Lemma.

**Lemma 1.** Let \( \{ \Pi_k \}_{k=1}^N \subset \mathbb{R}^{(n_k+n+\omega)} \times (n_k+n+\omega) \), \( \gamma > 0 \), and \( \{ \lambda_k \}_{k=1}^N \subset \mathbb{R} \) be given. Let each \( \Pi_k \) has a factorization \( (\Psi_k, M_k) \) where \( \Psi_k \) is stable. Define \( \Pi_\lambda := \sum_{k=1}^N \lambda_k S_k \forall \Pi_k S_k, \) and assume \( (\Pi_\lambda)_1(\omega) > 0 \) and \( (\Pi_\lambda)_2(\omega) < 0 \) \( \forall \omega \in \mathbb{R} \cup \{ +\infty \} \). Thus \( \Pi_\lambda \) has a rescaled \( J \)-spectral factorization \( (\Psi_\lambda, M_\lambda) \) as defined above. Let \( X \) denote the corresponding stabilizing solution to the ARE (Equation A.2) with \((A, B, Q_\lambda, S_\lambda, R_\lambda)\). Finally, assume the nominal system \( H_\rho \) is stable.

Then, using the extended system notation defined above, the symmetric matrix \( P = PT \) satisfies Equation 25 for all \( \rho \in \mathcal{P} \) if and only if \( \tilde{\rho} := P + [0 \ 0] \) \( \geq 0 \) satisfies Equation 26 for all \( \rho \in \mathcal{P} \).

**Proof.** See Appendix C. \( \square \)

This technical lemma is used to prove the following main result.

**Theorem 3.** Assume \( F_u(H_\rho, \Delta) \) is well posed for all \( \Delta \in \Delta_{1/\gamma}(\Pi_1, \ldots, \Pi_N) \). Then \( H_\rho \) achieves robust performance of level \( \gamma \) if there exists a matrix \( P = PT \in \mathbb{R}^{(n+\omega)} \times (n+\omega) \) and scalars \( \lambda_k \geq 0 \) such that:

(i) \( (P, \lambda, \gamma) \) satisfy the parameterized matrix inequality in Equation 25 for all \( \rho \in \mathcal{P} \)

(ii) The combined multiplier \( \Pi_\lambda := \sum_{k=1}^N \lambda_k S_k \forall \Pi_k S_k \) satisfies \( (\Pi_\lambda)_1(\omega) > 0 \) and \( (\Pi_\lambda)_2(\omega) < 0 \) \( \forall \omega \in \mathbb{R} \cup \{ +\infty \} \).

**Proof.** As described above, assumption (ii) is sufficient to ensure the combined multiplier has a rescaled \( J \)-spectral factorization \( (\Psi_\lambda, M_\lambda) \). Define \( \tilde{P} := P + [0 \ 0] \geq 0 \) where \( X \) is the stabilizing ARE solution used to construct this factorization. By Lemma 1, \( \tilde{P} \) satisfies the parameterized matrix in equality in Equation 26. The remainder of the proof is based on dissipation theory using the storage function \( V : R^{n+\omega} \to \mathbb{R} \) defined as \( V(x) := xT \tilde{P} x \). Left and right multiply Equation 26 by \([x^T, w^T, d^T]\) and \([x^T, w^T, d^T]^T\) to show that \( V \) satisfies the dissipation inequality:

\[
\dot{V}(t) + z(t) \lambda_k z(t) \leq d(t)^T d(t) - \gamma^{-2} e(t)^T e(t)
\]  

(27)

(Append \( \Psi_\lambda \) to the \((v, w)\) channels of the uncertain system \( F_u(H_\rho, \Delta) \). This corresponds to the interconnection shown in Figure 4 except with \( \Psi \), replacing \( \Psi_1/\gamma \). Let \((x, w, d, z, e)\) the solution of this interconnection for some \( \Delta \in \Delta_{1/\gamma}(\Pi_1, \ldots, \Pi_N), \) disturbance \( d \in L_2, \) admissible trajectory \( \rho \in \mathcal{T}, \) and zero initial conditions. Integrating the dissipation inequality (27) along this solution from \( t = 0 \) to \( t = T \) yields:

\[
V(x(T)) + \int_0^T z(t) \lambda_k z(t) dt + \frac{1}{\gamma^2} \int_0^T e(t)^T e(t) dt \leq \int_0^T d(t)^T d(t) dt
\]  

(28)

It follows from \( \lambda_k \geq 0 \) that \( \Delta \in \Delta(\Pi_\lambda) \). In addition, the rescaled \( J \)-spectral factorization \( (\Psi_\lambda, M_\lambda) \) is a valid factorization of \( \Pi_\lambda \) by Lemma 6. Therefore, \( (\Psi_\lambda, M_\lambda) \) is a valid time domain IQC for \( \Delta \). Apply this time domain IQC along with \( \tilde{P} \geq 0 \) to Equation 28 to conclude that \( \|e\| \leq \gamma \|d\| \). Hence \( H_\rho \) achieves RP of level \( \gamma \). \( \square \)

The parameterized matrix inequality (Equation 25) in condition (i) of Theorem 3 involves \( N \) IQCs. Note that left/right multiplying Equation 25 by \([x^T, w^T, d^T]\) and \([x^T, w^T, d^T]^T\) does not yield a true dissipation inequality for two reasons. First, \( (\Psi_\lambda, M_\lambda) \) does not need to be a hard factorization and hence not a valid time domain IQC by Definition 2. Second, the matrix \( P \) need not be positive definite and thus does not necessarily define a valid storage function. The technical result in Lemma 1 addresses both issues. It converts the original problem in an alternative form (equation 26) involving only a single, valid time domain IQC. The alternative form involves \( \tilde{P} = P + [0 \ 0] \geq 0 \) which defines a valid storage function. The term \( X \), which is the stabilizing solution of the ARE used to construct a hard factorization, can be interpreted as additional energy. Finally, it is important to recall that soft IQCs only hold, in general, over finite time horizons and they require the signals \((v, w)\) to be in \( L_2 \). Hence soft IQCs cannot be used in the dissipation inequality proof since we don’t know, a priori, that \((v, w)\) are in \( L_2 \). On the other hand, hard IQCs hold over all finite time horizons and for all signals \((v, w)\) in the extended space \( L_{2e} \). Hence a hard factorization of \( \Pi_\lambda \) is needed.
4.2 Scaled System

This section constructs a specific scaled system that will be used to link the analysis and synthesis steps in our robust (IQC) synthesis algorithm. Consider the uncertain system $F_u(H_\rho, \Delta)$. Theorem 3 provides a sufficient condition to ensure that $H_\rho$ achieves robust performance of level $\gamma$. The proof involves a rescaled $J$-spectral factorization $(\Psi_\Lambda, M_\Lambda)$. In particular, robust performance is shown via a dissipation inequality (Equation 27) defined for the extended system of $H_\rho$ and $\Psi_\Lambda$. Recall that the $J$-spectral factorization was rescaled so that $M_\Lambda := \begin{bmatrix} \gamma^{-2}I & 0 \\ 0 & -I \end{bmatrix}$. Thus partitioning $z_\Lambda := \begin{bmatrix} w_\Lambda \\ \tilde{w}_\Lambda \end{bmatrix}$ simplifies the dissipation inequality to

$$
\dot{V}(t) \leq (d(t)^T d(t) - \gamma^{-2}e(t)^T e(t)) + (w_\Lambda(t)^T w_\Lambda(t) - \gamma^{-2}v_\Lambda(t)^T v_\Lambda(t))
$$

The form of this dissipation inequality implies a connection to nominal induced $L_2$ gain performance. Note that Theorem 1 in Section 2.2 provides a sufficient condition to upper bound the induced $L_2$ gain of an LPV system. The proof for this nominal performance condition uses a dissipation inequality (Equation 4) that is similar to Equation 29. In particular, Equation 29 has the form of a dissipation inequality used to prove a (not-uncertain) LPV system with inputs $(w_\Lambda, d)$ and outputs $(v_\Lambda, e)$ has induced gain $\leq \gamma$. Based on this insight, a scaled system will be constructed with these inputs and outputs. First, rewrite the extended system of $H_\rho$ and $\Psi_\Lambda$ (Equation 24) by partitioning $z_\Lambda := \begin{bmatrix} w_\Lambda \\ \tilde{w}_\Lambda \end{bmatrix}$:

$$
\begin{bmatrix}
\dot{x} \\
v_\Lambda \\
w_\Lambda \\
e
\end{bmatrix} =
\begin{bmatrix}
A(p) & B_w(p) & B_d(p) \\
C_v(p) & D_{v,w}(p) & D_{v,d}(p) \\
C_w(p) & D_{w,w}(p) & D_{w,d}(p) \\
C_e & D_{e,w} & D_{e,d}(p)
\end{bmatrix}
\begin{bmatrix}
x \\
w \\
0
\end{bmatrix}
$$

(30)

Assume that $D_{w,w}(p)$ is nonsingular $\forall p \in P$. Then the output equation for $w_\Lambda$ can be rewritten as:

$$
w = D_{w,w}(p)^{-1}(w_\Lambda - C_{w,w}(p)x - D_{w,d}(p)d)
$$

(31)

Use this relation to substitute for $w$ in the extended system (Equation 30). This gives the following “scaled” system with inputs $(w_\Lambda, d)$ and outputs $(v_\Lambda, e)$ (neglecting dependence on $p$):

$$
\begin{bmatrix}
\dot{x} \\
v_\Lambda \\
e
\end{bmatrix} =
\begin{bmatrix}
A & B_w & B_d \\
C_v & D_{v,w} & D_{v,d} \\
C_e & D_{e,w} & D_{e,d}
\end{bmatrix}
\begin{bmatrix}
I & 0 & 0 \\
-D_{w,w}^{-1}C_w & D_{w,d}^{-1} - D_{w,w}^{-1}D_{w,d} & 0 \\
0 & 0 & I
\end{bmatrix}
\begin{bmatrix}
x \\
w_\Lambda \\
d
\end{bmatrix}
$$

(32)

The use of the term “scaled” system will be further clarified below. The next lemma gives a formal statement connecting robust performance of the extended system to nominal performance of this scaled system.

**Lemma 2.** Let $\bar{P} \geq 0$ and $\gamma > 0$ be given. The following statements are equivalent:

1. $(\bar{P}, \gamma)$ satisfy the robust performance LMI associated with the extended system of $H_\rho$ and $\Psi_\Lambda$ for all $p \in P$:

$$
\begin{bmatrix}
\bar{P}A_{ex} + A_{ex}^T \bar{P} & \bar{P}B_{ex} & \bar{P}B_{ex} \\
B_{ex}^T \bar{P} & 0 & 0 \\
B_{ex}^T \bar{P} & 0 & -I
\end{bmatrix} + \begin{bmatrix}
C_{ex}^T & C_{ex}^T \\
D_{ex,w}^T & D_{ex,w}^T \\
D_{ex,d}^T & D_{ex,d}^T
\end{bmatrix}
M_\Lambda
\begin{bmatrix}
C_{ex} & D_{ex,w} & D_{ex,d} \\
C_{ex} & D_{ex,w} & D_{ex,d} \\
C_{ex} & D_{ex,w} & D_{ex,d}
\end{bmatrix} + \frac{1}{\gamma^2}
\begin{bmatrix}
C_{ex}^T \\
D_{ex,w}^T \\
D_{ex,d}^T
\end{bmatrix}
\begin{bmatrix}
\bar{P}C_{ex} & \bar{P}D_{ex,w} & \bar{P}D_{ex,d} \\
\bar{P}C_{ex} & \bar{P}D_{ex,w} & \bar{P}D_{ex,d} \\
\bar{P}C_{ex} & \bar{P}D_{ex,w} & \bar{P}D_{ex,d}
\end{bmatrix}
< 0
$$

(33)

where the dependence on $p$ has been omitted.

2. $D_{w,w}(p)$ is nonsingular $\forall p \in P$. Let $(A_{ex}, B_{ex}, C_{ex}, D_{ex})$ denote the state-space representation of the scaled system formed from $H_\rho$ and $\Psi_\Lambda$ (Equation 32). $(\bar{P}, \gamma)$ satisfy the induced $L_2$ gain LMI (Equation 3) associated with the scaled system for all $p \in P$:

$$
\begin{bmatrix}
\bar{P}A_{ex} + A_{ex}^T \bar{P} & \bar{P}B_{ex} \\
B_{ex}^T \bar{P} & -I
\end{bmatrix} + \frac{1}{\gamma^2}
\begin{bmatrix}
C_{ex}^T \\
D_{ex}^T
\end{bmatrix}
\begin{bmatrix}
\bar{P}C_{ex} & \bar{P}D_{ex} \\
\bar{P}C_{ex} & \bar{P}D_{ex}
\end{bmatrix}
< 0
$$

(34)
Proof. (1 ⇒ 2) Assume statement 1 holds. The (2,2) block of Equation 33 implies:
\[
\gamma^{-2}D_{w_{\lambda}w}^TD_{w_{\lambda}w} - D_{w_{\lambda}w}^TD_{w_{\lambda}w} + \gamma^{-2}D_{cw}^TD_{cw} < 0
\]  
(35)

This inequality implies \(D_{w_{\lambda}w}^TD_{w_{\lambda}w} > \gamma^{-2}(D_{w_{\lambda}w}^TD_{w_{\lambda}w} + D_{cw}^TD_{cw}) \geq 0\) and hence \(D_{w_{\lambda}w}\) is nonsingular. Next, define the parameter-dependent congruence transformation:
\[
T(\rho) := \begin{bmatrix}
-I & 0 \\
0 & I
\end{bmatrix}
\]  
(36)

\(T\) is nonsingular for all \(\rho \in \mathcal{P}\). Multiplying Equation 33 on the left/right by \(T^T/T\) demonstrates that Equation 34 holds. The reverse implication \((2 ⇒ 1)\) follows by the inverse transformation. Specifically, multiply Equation 34 on the left/right by \(T^{-T}/T^{-1}\) to show that Equation 33 holds.

Multiplying the robust performance LMI in Equation 33 on the left/right by \([x^T, w^T, d^T]\) and its transpose yields the dissipation inequality in Equation 29. The congruence transformation \(T\) effectively changes to a dissipation inequality in variables \((x, w_{\lambda}, d)\). The lemma states that the robust performance condition for \(H_\rho\) is satisfied if and only if the nominal (induced \(L_2\) gain) performance condition is satisfied for the scaled system. The main issue at this point is that the extended system depends on \(H_\rho\) and \(\Psi_\lambda\). Thus the scaled system in Equation 32 appears to be a complicated function of the state matrices of \(H_\rho\) and \(\Psi_\lambda\). This is an issue because the robust synthesis algorithm will require the use of this result with the closed-loop, \(H_\rho := F_l(G_\rho, K_\rho)\).

In fact, the scaled system has a particularly simple construction. The extended system is formed by \(H_{\rho, \lambda}\) and \(\Psi_\lambda\). The scaled system is essentially formed by inverting the input/output channel associated with \(w\) to \(w_{\lambda}\). The channel from \(w\) to \(w_{\lambda}\) only involves the filter \(\Psi_\lambda\). The filter \(\Psi_\lambda\) (given in Equation 23) can be expressed in terms of the partitioned output \(z_{\lambda} := [w_{\lambda}]\) as:
\[
\begin{bmatrix}
\dot{x}_\psi \\
v_{\lambda} \\
w_{\lambda}
\end{bmatrix}
=
\begin{bmatrix}
\hat{A} & \gamma^{-1}\hat{B}_v & \hat{B}_w \\
\hat{C}_{v_{\lambda}} & \hat{D}_{v_{\lambda}v} & \hat{D}_{v_{\lambda}w} \\
0 & 0 & I
\end{bmatrix}
\begin{bmatrix}
x_\psi \\
v \\
w
\end{bmatrix}
\]  
(37)

The assumptions on the IQC multipliers for the robust synthesis algorithm (described in Section 3) will be sufficient to ensure that \(\hat{D}_{v_{\lambda}w}\) is nonsingular. If \(\hat{D}_{v_{\lambda}w}\) is nonsingular then \(w\) can be solved in terms of \((x_\psi, w_{\lambda}, v)\):
\[
w = \hat{D}_{v_{\lambda}w}^{-1} \left(w_{\lambda} - \hat{C}_{w_{\lambda}}x_\psi - \hat{D}_{w_{\lambda}w}v\right)
\]  
(38)

In this case, let \(\Psi_\lambda^I\) denote the filter from \((v, w_{\lambda})\) to \((v_{\lambda}, w)\) obtained by inverting the \(w\) to \(w_{\lambda}\) channel of \(\Psi_\lambda\). \(\Psi_\lambda^I\) has the following state-space realization:
\[
\begin{bmatrix}
\dot{x}_\psi(t) \\
v_{\lambda}(t) \\
w(t)
\end{bmatrix}
=
\begin{bmatrix}
\hat{A} & \gamma^{-1}\hat{B}_v & \hat{B}_w \\
\hat{C}_{v_{\lambda}} & \hat{D}_{v_{\lambda}v} & \hat{D}_{v_{\lambda}w} \\
0 & 0 & I
\end{bmatrix}
\begin{bmatrix}
x_\psi(t) \\
w_{\lambda}(t) \\
v(t)
\end{bmatrix}
\]  
(39)

The next lemma provides an alternative, but equivalent, construction for the scaled system as a simple linear fractional transformation.

Lemma 3. Assume \(\hat{D}_{v_{\lambda}w}\) is nonsingular so that \(\Psi_\lambda^I\) as defined in Equation 39 is well-defined. Moreover, assume \(\hat{D}_{v_{\lambda}w}(\rho)\) is nonsingular \(\forall \rho \in \mathcal{P}\) so that the scaled system formed from \(H_\rho\) and \(\Psi_\lambda\) (Equation 32) is well-posed. Then the scaled system is equivalent to the LFT interconnection of \(H_\rho\) and \(\Psi_\lambda^I\) as shown in Figure 5.
For the special case of LTI uncertainty the scaled system shown in Figure 5 reverts to that used in DK synthesis. As noted above, the use of $D$-scales in DK synthesis (for SISO LTI uncertainty) corresponds to the frequency domain IQC defined by $\Pi = \begin{bmatrix} \alpha & 0 \\ 0 & -\alpha \end{bmatrix}$ for any SISO, LTI system $\alpha$ such that $\alpha(j\omega) = \alpha(j\omega)^* > 0 \forall \omega \in \mathbb{R} \cup \{\infty\}$. Moreover, the rescaled $J$-spectral factorization in Step 6 is given by $M_\lambda := \begin{bmatrix} \gamma^{-1} & 0 \\ 0 & -I \end{bmatrix}$ and $\Psi_\lambda := \begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix}$ where $d$ is a spectral factor of $\alpha$. In this case, inverting the channels $w$ and $w_\lambda$ yields $\Psi_\lambda^T := \begin{bmatrix} 0 & d \\ -d^{-1} & 0 \end{bmatrix}$. The scaled system created by the LFT interconnection of $F_l(G_{scl}^{\rho}, K_\rho)$ and $\Psi_\lambda^T$ is thus given by $[d^{-1} \ 0 \ 0 \ 1] F_l(G_{scl}^{\rho}, K_\rho) [d^{-1} \ 0 \ 0 \ 1]$. This is precisely the standard scaled system that appears in DK synthesis.

### 4.3 Main Theorem

The main technical result for the proposed algorithm is that the iteration is well posed at each step and the robust performance is non-increasing at each iteration. Thus the closed-loop robust performance metric will eventually converge and the iteration in Algorithm 1 will terminate. As with DK synthesis, there are no guarantees that the coordinate-wise iteration will lead to a local optima let alone a global optima. However, the iteration is a useful heuristic that enables robust synthesis to extended naturally from LTI to LPV systems. This main convergence result is now stated.

**Theorem 4.** The iteration is well-posed at each step and the iteration is non-increasing, i.e. $\gamma(i) \leq \gamma(i-1)$ for $i = 1, 2, \ldots$.

**Proof.** Note that the initial iteration $i = 1$ slightly differs from the consecutive ones. Specifically, the choice of $\lambda(0) = [1, 0, \ldots, 0]$ yields $\Pi_{\lambda(0)} = \Pi_1$ in Step 6 of the first iteration. The definition of $\Pi_1$ (Assumption 4) implies that it has a simple $J$-factorization with $\Psi_1 := I_{n_w + n_w}$ and $M_1 := \begin{bmatrix} I_{n_w} & 0 \\ 0 & -I_{n_w} \end{bmatrix}$ in Step 7. No rescaling is used on the first iteration. The static filter $\Psi_1$ is equivalent to $z_\lambda := [w_\lambda]$ satisfying $v_\lambda = v$ and $w_\lambda = w$. In this case, the scaled system in Step 7 is simply $G_{scl}^{\rho} = G_\rho$. The synthesis step 8 is then performed with no special modifications for the initial step. As a result, the synthesis step 8 yields a controller $K_\rho(1)$ that stabilizes the system $G_\rho$ and achieves a finite closed-loop gain $\nu(1) < \infty$. This follows because the nominal system $G_\rho$ is quadratically stabilizable and detectable (Assumption 1). The analysis step of the first iteration then achieves a finite robust performance $\gamma(1) < \infty$ because the closed-loop $H_\rho$ is stable. Thus the first iteration is well-posed and achieves $\gamma(1) < \gamma(0) = +\infty$.

Subsequent iterations ($i > 1$) begin with the iteration count update (Step 4) and performance scaling definition (Step 5). Next the combined multiplier $\Pi_i$ is constructed. The coefficients from the previous analysis step satisfy $\lambda_i(i-1) \geq 0$ and $\lambda_i(i-1) > 0$. This fact along with Assumptions 3 and 4 imply that the combined multiplier satisfies $(\Pi_{\lambda(i-1)} j\omega) > 0$ and $(\Pi_{\lambda(i-1)})_2(j\omega) < 0 \forall \omega \in \mathbb{R} \cup \{\infty\}$. Thus the combined multiplier satisfies the sufficient conditions in Lemma 5 for the existence of a $J$-spectral factorization. In addition, $(\Pi_{\lambda(i-1)})_2(+\infty) < 0$ implies that the feedthrough matrix of $\Psi_\lambda$ from $w$ to $w_\lambda$ must be non-singular. In the notation of Section 4.2, this corresponds to nonsingularity of $\bar{D}_{w_\lambda w}$. Hence by Lemma 3, the construction of $\Psi_\lambda^T$ in Step 7 is well-defined.

The analysis step from the previous iteration involves the robust performance parameterized matrix inequality (Equation 25) with the factorized IQC multipliers $\{(\Psi_k, M_k)\}_{k=1}^N$. Hence there exists $(P(i-1), \lambda(i-1), \gamma(i-1))$...
satisfying Equation 25. By Lemma 1, this implies the existence of $\tilde{P}(i-1) \geq 0$ that, along with $(\lambda(i-1), \gamma(i-1))$, satisfies the matrix inequality (Equation 25) with the rescaled $J$-spectral factorization.

Next, Lemma 2 states that feasibility of Equation 25 (which is simply Equation 33 written in different notation) implies that the scaled closed-loop of $H_\rho := F_i(G_\rho, K_\rho(i-1))$ and $\Psi_\lambda$ is well-posed and has induced gain $\leq \gamma(i-1)$. By Lemma 3, this scaled system can be represented by the feedback interconnection of $H_\rho$ and $\Psi_\lambda^\dagger$ as shown in Figure 5. Removing the controller, i.e. opening up the $u/y$ channels, yields the scaled open-loop plant. Thus the construction of the scaled system in Step 7 is well-defined.

Finally, the synthesis in Step 8 optimizes over all stabilizing controllers. Hence the new controller $K_\rho(i)$ must yield a cost no greater than that achieved by the previous controller $K_\rho(i-1)$ on the scaled plant. Hence $\nu(i) \leq \gamma(i-1)$. Thus the new controller must satisfy the nominal performance LMI in Equation 34 with the slightly larger cost of $\gamma := \gamma(i-1)$. Lemmas 2 and 1 can be used to work backward to the analysis condition in Step 9. Specifically, the closed-loop with new controller $K_\rho(i)$ satisfies the analysis condition in Step 9 with the previous performance level $\gamma(i-1)$, scalings $\lambda(i-1)$ and matrix $P(i-1)$. Step 9 involves optimizing over all feasible coefficients $\lambda$ and matrix $P$. This must yield a robust performance cost no greater than the previous step $\gamma(i) \leq \gamma(i-1)$. □

5 Numerical Example

A simple example is used to demonstrate the applicability of the proposed robust synthesis algorithm. The example is based on an example that appears in [25] to test an alternative IQC synthesis algorithm for LTI systems. Here the example is extended to include plant dynamics described by an LPV system. The objective of the example is to design an LPV robust controller for the feedback system shown in Figure 6. The nominal plant dynamics are given by the following 2-input, 2-output LPV system $F_\rho$:

$$
\dot{x}(t) = \left( -\frac{1}{71+2\rho} I_2 \right) x(t) + \left( \frac{1}{\rho} I_2 \right) u(t)
$$

$$
y(t) = \left[ \begin{array}{cc} 87+0.2\rho^2 & -87.2+0.2\rho^2 \\ 107.4+0.2\rho^2 & -110.4+0.2\rho^2 \end{array} \right] x(t)
$$

The plant dynamics depend on a single scheduling parameter $\rho$ that is restricted to the interval $[1, 3]$. For all the following scenarios a grid of 5 points is used that span the parameter space equidistantly. This nominal LPV plant $F_\rho$ was constructed by modifying an LTI model for the idealized distillation process in [23]. The uncertainty $\Delta$ that appears Figure 6 is assumed to be a block diagonal, i.e. $\Delta := \begin{bmatrix} \Delta_i & 0 \\ 0 & \Delta_i \end{bmatrix}$. In addition, each block is assumed to be a dead zone operator $w_i = \Delta_i(v_i)$ defined by:

$$
w_i = \Delta_i(v_i) := \begin{cases} 
    v_i - b_i, & v_i > b_i \\
    0, & v_i \in [-b_i, b_i] \\
    v_i + b_i, & v_i < -b_i
\end{cases}
$$

where $b_i = 0.05, i = 1, 2$. The weight on $\Delta$ is defined as $W_d := \begin{bmatrix} 1.0 & 0 \\ 0 & 0.3 \end{bmatrix}$. It should be noted that classical DK synthesis is unable to solve a problem with nonlinear perturbation such as the dead zone introduced here.

The objectives of the robust LPV controller $K_{rob}$ are to offer good tracking performance at low frequencies while penalizing control input at high frequencies. These tracking and control effort performance objectives are specified

![Fig. 6. Synthesis interconnection](image-url)
respectively. As expected, the robust design achieves better nominal performance as measured with the $K_{\text{nom}}$ and $K_{\text{rob}}$ respectively. As expected, $K_{\text{nom}}$ achieves better performance, as measured with the $L_2$ norm bound. The induced $L_2$ norm of the nominal system using $K_{\text{nom}}$ and $K_{\text{rob}}$ is given by 0.99 and 1.15, respectively. As expected, $K_{\text{nom}}$ achieves better performance, as measured with the $L_2$ norm bound. Next, the robust performance of the closed-loop system was assessed using the matrix inequality conditions in Section 4.1. This yields 6.96 and 1.43 for $K_{\text{nom}}$ and $K_{\text{rob}}$, respectively. As expected, the robust design $K_{\text{rob}}$ achieves better robust performance.

The gap in robust performance between the two controllers is also illustrated by a time domain step response simulation (Figure 7). In the simulation, unit step signals are injected into both channels of $d$ simultaneously at $t = 10 \text{s}$ and the parameter trajectory is given by $\rho(t) = \sin(0.05 t) + 2$. The responses of $y_1$ and $y_2$ are shown in Figure 7. It is seen that $K_{\text{nom}}$ performs well (solid blue curve) when there is no uncertainty in the system. However, the gap in robust performance between the two controllers is also illustrated by a time domain step response simulation (Figure 7). In the simulation, unit step signals are injected into both channels of $d$ simultaneously at $t = 10 \text{s}$ and the parameter trajectory is given by $\rho(t) = \sin(0.05 t) + 2$. The responses of $y_1$ and $y_2$ are shown in Figure 7. It is seen that $K_{\text{nom}}$ performs well (solid blue curve) when there is no uncertainty in the system. However, it degrades dramatically (dash-dot red curve) when the uncertainty is added. In contrast, $K_{\text{rob}}$ maintains good tracking and steady state error (dash green curve) with existence of the uncertainty.

6 CONCLUSION

This paper described a robust synthesis algorithm for a class of uncertain LPV systems. The proposed algorithm involves a coordinate-wise iteration between an LPV synthesis step and an IQC analysis step. It was shown that the closed-loop robust performance is a non-increasing function of the iteration number. The effectiveness of this method was shown on a simple numerical example. Future work will consider refinements of the proposed algorithm including a more efficient parameterization of the IQC multipliers. In addition, the algorithm will be applied to design a robust LPV controller for a more realistic system.

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References

Fig. 7. Step responses


A IQC Factorizations

Two factorizations of the frequency domain IQC multiplier $\Pi$ are provided in Lemma 4 and Lemma 5. The first factorization given by Lemma 4 only assumes $\Pi$ to be rational and uniformly bounded. However, this factorization $(Ψ, M)$ does not, in general, yield a valid time domain IQC. The second factorization given by Lemma 5 is called a $J$-spectral factorization and it requires additional assumptions on $\Pi$. Lemma 6 shows that this $J$-spectral factorization yields a valid time domain IQC.

**Lemma 4.** If $\Pi = \Pi^{-}\in \mathbb{R}_{s}^{m\times m}$ then there exists real matrices $\hat{A}, \hat{B}, \hat{Q}, \hat{S}, \hat{R}$ of compatible dimensions with $\hat{A}$ Hurwitz, $\hat{Q} = \hat{Q}\top$, and $\hat{R} = \hat{R}\top$ such that

$$\Pi(s) = \begin{bmatrix} \hat{B} & (-sI - \hat{A}\top)^{-1}I \end{bmatrix} \begin{bmatrix} \hat{Q} & \hat{S} \end{bmatrix} \begin{bmatrix} (sI - \hat{A})^{-1}\hat{B} \\ I \end{bmatrix}$$ (A.1)

**Proof.** The proof follows from arguments in Section 7.3 of [8]. Let $(\hat{A}_x, \hat{B}_x, \hat{C}_x, \hat{D}_x)$ be a minimal state-space realization for $\Pi$. Separate $\Pi$ into its stable and unstable parts $\Pi = G_S + G_U + D_x$. Let $(\hat{A}, \hat{B}, \hat{C}, 0)$ denote a state-space realization for the stable part $G_S$. The assumptions on $\Pi$ imply that $G_U$ has a state-space realization of the form $(-\hat{A}\top, -\hat{C}\top, \hat{B}\top, 0)$ (Section 7.3 of [8]). Thus $\Pi = G_S + G_U + D_x$ can be written as in Equation A.1 with $\hat{Q} = 0$, $\hat{S} = \hat{C}\top$ and $\hat{R} = \hat{D}_x$.

Lemma 4 provides a factorization of $\Pi$ in the form $Ψ^{-}MΨ$ where $Ψ(s) := \begin{bmatrix} (sI - \hat{A})^{-1}\hat{B} \\ I \end{bmatrix}$ and $M := \begin{bmatrix} \hat{Q} & \hat{S} \\ \hat{S}\top & \hat{R} \end{bmatrix}$. The main construction in the proof is to separate the stable and unstable parts of $\Pi$. This can be easily implemented in Matlab using the *stabsep* command. This provides a factorization $\Pi = Ψ^{-}MΨ$ where $Ψ \in \mathbb{R}_{s}^{m\times m}$ is stable but non-square. Moreover, this factorization $(Ψ, M)$ does not, in general, yield a valid time domain IQC as described in Definition 2. Lemma 5 below states another special factorization with some additional assumptions on $\Pi$.

**Lemma 5.** Let $\Pi = \Pi^{-}\in \mathbb{R}_{s}^{n_{w}\times m}$ be partitioned as $\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12} & \Pi_{22} \end{bmatrix}$ where $\Pi_{11} \in \mathbb{R}_{s}^{n_u\times n_v}$ and $\Pi_{22} \in \mathbb{R}_{s}^{n_w\times n_w}$. If $\Pi_{11}(j\omega) > 0$ and $\Pi_{22}(j\omega) < 0 \forall \omega \in \mathbb{R} \cup \{\infty\}$, then

(i) There exists real matrices $\hat{A}, \hat{B}, \hat{Q}, \hat{S}, \hat{R}$ of compatible dimensions with $\hat{A}$ Hurwitz, $\hat{Q} = \hat{Q}\top$, and $\hat{R} = \hat{R}\top$ such that $\Pi$ can be expressed as in Equation A.1.

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(ii) \( \tilde{R} \) is nonsingular and there exists a unique real solution \( X = X^T \) to the the following ARE

\[
\tilde{A}^T X + X \tilde{A} - (X \tilde{B} + \tilde{S}) \tilde{R}^{-1}(X \tilde{B} + \tilde{S})^T + \tilde{Q} = 0 \quad (A.2)
\]

such that \( \tilde{A} - \tilde{B} \tilde{R}^{-1}(X \tilde{B} + \tilde{S})^T \) is Hurwitz.

(iii) \( \Pi \) has a factorization \((\Psi, M)\) with \( M := \begin{bmatrix} I_{nv} & 0 \\ 0 & -I_{nw} \end{bmatrix} \) and \( \Psi, \Psi^{-1} \in \mathbb{R}^{(n_v + n_w) \times (n_v + n_w)} \). A state-space realization of \( \Psi \) is given by \( \left( \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} \right) \) where \( \tilde{D} \) is a solution of \( \tilde{R} = \tilde{D}^T M \tilde{D} \) and \( \tilde{C} := M \tilde{D}^T \left( \tilde{B}^T X + \tilde{S}^T \right) \).

**Proof.** Conclusion (i) holds for any \( \Pi = \Pi^\sim \) and follows from Lemma 4. Conclusions (ii) and (iii) follow from Lemma 4 in [22].

The factorization in Conclusion (iii) is called a \( J \)-spectral factorization of \( \Pi \). For this factorization, \( \Psi \) is square, stable, and stably invertible. Existence conditions for a \( J \)-spectral factor of \( \Pi \) are provided by the canonical factorization theorem in [3]. Chapter 7 of [8] summarizes these results. Existence conditions for a \( J \)-spectral factor can also be specified using the notion of an equalizing vector as defined in [14]. Lemma 5 above provides an alternative existence condition for a \( J \)-spectral factorization in terms of definiteness properties on \( \Pi \). Lemma 6 below states that the \( J \)-spectral factorization is a hard factorization of \( \Pi \). Thus a frequency domain IQC multiplier can, under some additional assumptions on \( \Pi \), be factorized to yield a valid time domain IQC. Lemma 6 also provides an additional technical result that will be used in this paper.

**Lemma 6.** Let \( \Pi = \Pi^\sim \in \mathbb{R}^{(n_v + n_w) \times (n_v + n_w)} \) be partitioned as \( \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^\top & \Pi_{22} \end{bmatrix} \) where \( \Pi_{11} \in \mathbb{R}^{n_v \times n_v} \) and \( \Pi_{22} \in \mathbb{R}^{n_w \times n_w} \). Assume \( \Pi_{11}(j\omega) > 0 \) and \( \Pi_{22}(j\omega) < 0 \) \( \forall \omega \in \mathbb{R} \cup \{\infty\} \). Let \( (\Psi, M) \) be the \( J \)-spectral factorization given in Conclusion (iii) of Lemma 5. Then,

(i) \( (\Psi, M) \) is a hard factorization of \( \Pi \): If \( \Delta \) is a bounded, causal operator that satisfies the frequency domain IQC specified by \( \Pi \) (Definition 1) then \( \Delta \) satisfies the time domain IQC specified by \( (\Psi, M) \) (Definition 2).

(ii) The cost of the max/min game defined in Equation A.3 based on \( (\Psi, M) \) satisfies \( J(x_{\psi0}) = 0 \).

**Proof.** Conclusion (i) follows from Theorem 2.4 in [12] or Theorem 4 in [22]. Conclusion (ii) follows from Lemma 5 and the proof of Theorem 4 in [22]. \( J \) is the lower-value of a two-player differential game and the proof of Conclusion (ii) given in [22] essentially relies on results for LQ differential games [6,7].

\( J(x_{\psi0}) := \sup_{w \in L^2_{2\pi}[0,\infty]} \inf_{v \in L^2_{2\pi}[0,\infty]} \int_0^\infty z(t)^T M z(t) \, dt \quad (A.3) \)

subject to:
\[
\dot{x}_\psi = \tilde{A} x_\psi + \tilde{B}[v] \quad , \quad x_\psi(0) = x_{\psi0}
\]
\[
z = \tilde{C} x_\psi + \tilde{D}[v]
\]
B Extended System State Matrices

A state-space realization for the extended system of \( H_p \) and \( \Psi_{1/\gamma} \) is given in Equation 10. The state matrices for the extended system can be expressed in terms of the state matrices for \( H_p \) (Equation 17) and \( \Psi_{1/\gamma} \) (Equation 18) as:

\[
A(\rho) := \begin{bmatrix} A(\rho) & 0 \\ \gamma^{-1}B_eC_v(\rho) & \tilde{A} \end{bmatrix}
\]  \hfill (B.1)

\[
B_w(\rho) := \begin{bmatrix} B_w(\rho) \\ \gamma^{-1}B_eD_wu(\rho) + \tilde{B}_w \end{bmatrix}, \\
B_d(\rho) := \begin{bmatrix} B_d(\rho) \\ \gamma^{-1}B_eD_{vd}(\rho) \end{bmatrix}
\]  \hfill (B.2)

\[
C_{z_k}(\rho) := \begin{bmatrix} \gamma^{-1}\tilde{D}_{z_k}C_v(\rho), \tilde{C}_{z_k} \end{bmatrix}, \\
C_e(\rho) := \begin{bmatrix} C_e(\rho), 0 \end{bmatrix}
\]  \hfill (B.3)

\[
D_{z_k,w}(\rho) := \gamma^{-1}\tilde{D}_{z_k}D_wu(\rho) + \tilde{D}_{z_k,w}, \\
D_{e,w}(\rho) := D_{e,w}(\rho), \\
D_{e,d}(\rho) := D_{e,d}(\rho)
\]  \hfill (B.4)

\[
D_{z_k,d}(\rho) := \gamma^{-1}\tilde{D}_{z_k}D_{vd}(\rho)
\]  \hfill (B.5)

Similarly, the state-space realization for the extended system of \( H_p \) and \( \Psi_\lambda \) is given in Equation 24. These state matrices can be expressed in terms of the state matrices for \( H_p \) (Equation 17) and \( \Psi_\lambda \) (Equation 23). Only the output and feedthrough matrices associated with the output \( z_\lambda \) are changed. These are given by:

\[
C_{z_k}(\rho) := \begin{bmatrix} \gamma^{-1}\tilde{D}_{z_k}C_v(\rho), \tilde{C}_{z_k} \end{bmatrix}
\]  \hfill (B.6)

\[
D_{z_k,w}(\rho) := \gamma^{-1}\tilde{D}_{z_k}D_wu(\rho) + \tilde{D}_{z_k,w}
\]  \hfill (B.7)

\[
D_{z_k,d}(\rho) := \gamma^{-1}\tilde{D}_{z_k}D_{vd}(\rho)
\]  \hfill (B.8)

C Proof of Lemma 1

Proof. (\( \Rightarrow \)) Assume \( P = P^T \) satisfies Equation 25. The output \( z_k \) from \( \Psi_{1/\gamma} \) is a linear function of \( (x_w,v,w) \) as defined in Equation 18:

\[
z_k = [C_{z_k} \gamma^{-1}\tilde{D}_{z_k}v, \tilde{D}_{z_k}w] \begin{bmatrix} x_w \\ v \\ w \end{bmatrix}
\]  \hfill (C.1)

These variables \( (x_w,v,w) \) can, in turn, be expressed in terms of the extended system state and inputs \( (x,w,d) \) as:

\[
\begin{bmatrix} x_w \\ v \\ w \end{bmatrix} = \begin{bmatrix} [0, I] & 0 & 0 \\ [C_e(\rho), 0] & D_{e,w}(\rho) & D_{e,d}(\rho) \\ [0, 0] & I & 0 \end{bmatrix} \begin{bmatrix} x_H \\ y_g \\ \rho \end{bmatrix} = L(\rho) \begin{bmatrix} x_H \\ y_g \\ \rho \end{bmatrix}
\]  \hfill (C.2)

Thus, using the extended system state matrices defined in Appendix B, the second term of the matrix inequality in Equation 25 can be rewritten as:

\[
\sum_{k=1}^{N} \lambda_k \begin{bmatrix} C_{z_k}^T & D_{z_k}^T & D_{z_k,d}^T \end{bmatrix} M_k \begin{bmatrix} C_{z_k} & D_{z_k} & D_{z_k,d} \end{bmatrix} = L(\rho)^T \hat{Q}_\lambda \hat{S}_\lambda \hat{R}_\lambda L(\rho)
\]  \hfill (C.3)

\( \hat{Q}_\lambda, \hat{S}_\lambda, \) and \( \hat{R}_\lambda \) are defined in Equation 22. Substitute for \( \hat{Q}_\lambda \) using the ARE in Equation A.2. Rearrange terms in the matrix inequality to show that \( \hat{P} \) satisfies Equation 26.

This direction of the proof is completed by showing that \( \hat{P} \geq 0 \). Define the quadratic function \( V(x_0) := x_H^T \hat{P} x_0 \). In addition, define the following quadratic cost functional \( V^*(x_0) \) based on the extended system of \( H_p \) and the rescaled
factorization \((\Psi_\lambda, M_\lambda)\):

\[
V^*(x_0) := \sup_{w \in L^2_{2w}[0, \infty)} \int_0^\infty z_\lambda(t)^T M_\lambda z_\lambda(t) \, dt \tag{C.4}
\]

subject to:
\[
\dot{x} = A(\rho)x + B(\rho)w, \quad x(0) = x_0
\]
\[
z_\lambda = C_{z_\lambda}(\rho)x + D_{z_\lambda}(\rho)w
\]

The disturbance input of the extended system is neglected \(d = 0\) in this linear quadratic optimization. Note that the extended system is stable since \(H_p\) is stable (by assumption), \(\Psi_\lambda\) is stable (by construction), and \(\Psi_\lambda\) is connected in an open loop fashion to \(H_p\). First we show that \(V(x_0) \geq V^*(x_0)\) for all \(x_0 \in \mathbb{R}^{n_H+n_0}\). This follows along the lines of Theorems 2 and 3 in [28] and hence the proof is only sketched. Let \(x(t), z_\lambda(t)\) be the resulting solutions of the extended system of \(H_p\) and \(\Psi_\lambda\) for a given input \(w \in L_2^{2w}[0, \infty)\), admissible trajectory \(\rho \in T\), and initial condition \(x_0 \in \mathbb{R}^{n_H+n_0}\) assuming \(d = 0\). Multiply the matrix inequality in Equation 26 on the left/right by \([x(t) \ w(t)]^T\) and \([x(t) \ w(t)]\) to show \(\dot{V}(x(t)) + z_\lambda(t)^T M_\lambda z_\lambda(t) \leq 0\). Integrate this inequality from \(t = 0\) to \(t = T\) to obtain

\[
V(x(T)) + \int_0^T z_\lambda(t)^T M_\lambda z_\lambda(t) \, dt \leq V(x_0) \tag{C.5}
\]

\[
\lim_{T \to \infty} x(T) = 0
\]

for any \(w \in L_2^{2w}[0, \infty)\) because the extended system is stable. Maximizing the left side of Equation C.5 over \(w \in L_2^{2w}[0, \infty)\) for \(T = \infty\) thus yields \(V(x_0) \geq V^*(x_0)\).

Next, consider the max/min game defined for the rescaled \(J\)-spectral factorization \((\Psi_\lambda, M_\lambda)\):

\[
J(x_{\psi0}) := \sup_{w \in L_2^{2w}[0, \infty)} \inf_{v \in L_2^{2v}[0, \infty]} \int_0^\infty z_\lambda(t)^T M_\lambda z_\lambda(t) \, dt \tag{C.6}
\]

subject to:
\[
\dot{x}_\psi = \tilde{A}x_\psi + \tilde{B}[w], \quad x_\psi(0) = x_{\psi0}
\]
\[
z = \tilde{C}_{z_\lambda}x_\psi + \tilde{D}_{z_\lambda}[w]
\]

where \(\tilde{D}_{z_\lambda} := [\tilde{D}_{z_\lambda}, \tilde{D}_{z_\lambda}w]\). This max/min game is connected to the quadratic optimization defined in Equation C.4. Specifically, restricting \(v\) in the max/min game to be the output of \(H_p\) generated by \(w \in L_2\), \(d = 0\), and \(x_{H0}(0) = x_{H0}\) yields the quadratic optimization in Equation C.4. This specific choice of \(v\) yields a value that is no lower than the infimum over all possible \(v \in L_2\). Hence the max/min game yields the bound \(J(x_{\psi0}) \leq V^*(x_0)\). By Lemma 6, the cost of this max/min game is \(J(x_{\psi0}) = 0\). Putting these results together yields the following inequality

\[
0 = J(x_{\psi0}) \leq V^*(x_0) \leq V(x_0) := x_0^T \tilde{P} x_0 \tag{C.7}
\]

This holds for any \(x_0\) and thus \(\tilde{P} \geq 0\).

\((\Leftarrow)\) This direction of the proof essentially involves reversing the algebraic rearrangement to go from the matrix inequality in Equation 26 to the form in Equation 25.