Second-order Step Response

\[ y' + a_1 y' + a_2 y = b u \]

For simplicity, we'll consider zero initial conditions and a unit step input:

\[
\begin{align*}
\text{zero:} & \quad y(0) = 0 \\
\text{IC:} & \quad y(0) = 0 \\
\text{unit:} & \quad u(t) = \begin{cases} 
0 & t < 0 \\
1 & t \geq 0
\end{cases}
\]

We'll assume that the system is stable, i.e., the both roots of the characteristic equation have negative real parts and hence the free response decays to zero for any initial condition.

Recall that the roots of the characteristic equation are:

\[ r = -\frac{a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} \]

You can show that both roots have negative real part if and only if \( a_1 > 0 \), \( a_2 > 0 \). Thus for second-order systems, stability is equivalent to \( a_1 > 0 \), \( a_2 > 0 \).

The formulas will work out cleaner (and align with standard notation in the textbook) if we redefine the ODE coefficients as:

- \( a_1 = 2f_wn \)
- \( a_2 = \omega_n^2 \)

\[ f = \frac{a_1}{2f_wn} = \frac{a_1}{2\sqrt{a_2}} \]

\[ \omega_n = \sqrt{a_2} \]

Since we are assuming \( a_1 > 0 \), \( a_2 > 0 \), it follows that \( f \) and \( \omega_n \) are also \( > 0 \), and

\( \omega_n \) is called the natural frequency

\( f \) is called the damping ratio.
In terms of the new parameters $\tau, w_\text{n}$:

$$\ddot{y} + 2\tau w_\text{n} \dot{y} + w_\text{n}^2 y = bu$$
$$y(0) = 0, \quad \dot{y}(0) = 0, \quad u(t) = \begin{cases} 1 & \tau \geq 0 \\ 0 & \tau < 0 \end{cases}$$

We can derive the response to a unit-step input using the method of homogeneous and particular solutions.

Step 1) Find all homogeneous solutions to:

$$\ddot{y} + 2\tau w_\text{n} \dot{y} + w_\text{n}^2 y = 0$$

As shown in the derivation of the free response, each is a homogeneous solution if $\tau$ satisfies the characteristic polynomial:

$$\tau^2 + (2\tau w_\text{n}) \tau + (w_\text{n}^2) \tau = 0$$

With the new variables $(\tau, w_\text{n})$ the roots are given by

$$\tau = \frac{-2\tau w_\text{n} \pm \sqrt{4\tau^2 w_\text{n}^2 - 4w_\text{n}^2}}{2}$$

$$\Rightarrow \quad \tau = -\tau w_\text{n} \pm w_\text{n} \sqrt{\tau^2 - 1}$$

There are 3 cases to consider:

a) $\tau > 1$ : Overdamped
b) $\tau = 1$ : Critically Damped
c) $0 < \tau < 1$ : Underdamped

In deriving the free response solution we combined cases a) + b), however, these they have slightly different solutions.
a) $p>1$: Overdamped

The roots are:
\[ r_1 = -fw_n + \omega_n \sqrt{p^2 - 1} \]
\[ r_2 = -fw_n - \omega_n \sqrt{p^2 - 1} \]

All homogeneous solutions have the form
\[ y_H(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \]

b) $p=1$: Critically Damped

The roots are repeated:
\[ r_1 = r_2 = -fw_n = -\omega_n \]

All homogeneous solutions have the form
\[ y_H(t) = c_1 e^{r_1 t} + c_2 t e^{r_2 t} \]

where \( r = -\omega_n \).

c) $0<p<1$: Underdamped

The roots are a complex pair
\[ r_{1,2} = -fw_n \pm j\omega_n \sqrt{1-p^2} \]

All homogeneous solutions have the form
\[ y_H(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \]

As in the free-response solution (see p42) we can recombine terms to remove the complex exponents and obtain a real solution
\[ y_H(t) = e^{-fw_n t} \left[ A \cos \left( \omega_n \sqrt{1-p^2} t \right) + B \sin \left( \omega_n \sqrt{1-p^2} t \right) \right] \]

where $A$ and $B$ are to be determined.

Note that $y_H(t)$ has a decaying exponential $e^{-fw_n t}$ in the amplitude and sinusoids oscillating at frequency $\omega_n \sqrt{1-p^2}$.
Step 2) Find a particular solution to
\[ y^2 + 2f w_n y + w_n^2 y = b u(t) \]

Since \( y(t) = 1 \) for \( t < 0 \), a particular solution is
given by \( y(t) = \frac{b}{w_n^2} \) for \( t < 0 \).

Step 3) The solution is given by
\[ y(t) = y_p(t) + y_H(t) \]

Use the initial conditions \( y(0) = y'(0) = 0 \) to solve
for the unknown coefficients in \( y_H(t) \).

The solutions for the 3 cases are:

a) \( p > 1 \): Overdamped

\[ y(t) = \frac{b}{w_n^2} + C_1 e^{r_1 t} + C_2 e^{r_2 t} \]

\[ \begin{align*}
   y(0) &= 0 = C_1 + C_2 + \frac{b}{w_n^2} \\
   y'(0) &= 0 = C_1 r_1 + C_2 r_2 
\end{align*} \]

\[ \begin{align*}
   C_1 &= \frac{b r_1}{(r_1 - r_2) w_n^2} \\
   C_2 &= \frac{b r_2}{(r_2 - r_1) w_n^2} 
\end{align*} \]

b) \( p = 1 \): Critically damped

\[ y(t) = \frac{b}{w_n^2} + C_1 e^{r t} + C_2 t e^{r t} \]

\[ \begin{align*}
   y(0) &= 0 = C_1 + \frac{b}{w_n^2} \\
   y(t) &= 0 = C_1 r + C_2 
\end{align*} \]

\[ \begin{align*}
   C_1 &= -\frac{b}{w_n^2} \\
   C_2 &= \frac{b}{w_n^2} r = -\frac{b}{w_n} 
\end{align*} \]

\[ y(t) = \frac{b}{w_n^2} \left[ 1 - e^{-w_n t} - w_n t e^{-w_n t} \right] \]
c) \( y < 1 \): Underdamped

\[
y(t) = \frac{b}{w_n^2} + e^{-\xi w_n t} \left[ A \cos(\frac{\omega_n \sqrt{1-\xi^2}}{\omega_d} t) + B \sin(\frac{\omega_n \sqrt{1-\xi^2}}{\omega_d} t) \right]
\]

\[
\begin{align*}
y(0) &= 0 = \frac{b}{w_n^2} + A \\
\dot{y}(0) &= 0 = -\xi w_n A + B w_n \sqrt{1-\xi^2} \\
\end{align*}
\]

\[
\Rightarrow \ y(t) = \frac{b}{w_n^2} \left[ 1 - e^{-\xi w_n t} \left( \cos(\frac{\omega_n \sqrt{1-\xi^2}}{\omega_d} t) + \frac{\xi}{\omega_d} \eta \sin(\frac{\omega_n \sqrt{1-\xi^2}}{\omega_d} t) \right) \right]
\]

\[\text{Ex)} \ \ddot{y} + 4\dot{y} + 13y = 2u(t)\]

\[
y(0) = 0, \ \dot{y}(0) = 0, \ u(t) = \begin{cases} 0 & t < 0 \\
1 & t \geq 0
\end{cases}
\]

The characteristic equation is: \( r^2 + 4r + 13 = 0 \)

\[
\Rightarrow \ r = -2 \pm 3j
\]

\[
\text{Note: } \omega_n = \sqrt{13}
\]

\[
2\xi \omega_n = 4 \rightarrow \xi = \frac{\sqrt{13}}{2} < 1 \quad (\text{The system is underdamped.})
\]

The solution is of the form

\[
y(t) = \frac{2}{13} + e^{-2t} \left[ A \cos(3t) + B \sin(3t) \right]
\]

Using the initial conditions to solve for \( A \) and \( B \)

\[
y(0) = 0 = \frac{2}{13} + A \Rightarrow A = -\frac{2}{13}
\]

\[
\dot{y}(0) = 0 = -2A + 3B \Rightarrow B = \frac{4}{3} A = \frac{4}{39}
\]

\[
y(t) = \frac{2}{13} + e^{-2t} \left[ (-\frac{2}{13}) \cos(3t) + (-\frac{4}{39}) \sin(3t) \right]
\]

Notice that the response "overshoots" the steady-state value.
Summary of Second-order Step Response

\[ y + 2\zeta \omega_n y + \omega_n^2 y = b u \]
\[ y(0) = 0, \quad y(0) = 0, \quad u(t) = U_0 \quad t \geq 0 \]

When we studied first-order systems, we were mainly concerned with the speed of response (measured by the time constant) and the final steady-state value.

For second-order systems, it is again easy to find the steady-state value. If the system is stable, then the step response will reach a steady-state characterized by \( y = y' = 0 \). Thus, the steady-state value of \( y \) is given by:

\[ \omega_n^2 y_{ss} = b u \Rightarrow y_{ss} = \frac{b}{\omega_n^2} \]

Measuring the speed of response is more complicated, due to \( \zeta \).

First consider the overdamped case (\( \zeta > 1 \)). In this case, the response is:

\[ y(t) = \frac{b}{\omega_n^2} + C_1 e^{\zeta_1 t} + C_2 e^{\zeta_2 t} \]

If \( |\zeta_1| < |\zeta_2| \), then \( e^{\zeta_1 t} \) will decay much faster than \( e^{\zeta_2 t} \). In this case, the response is approximately:

\[ y(t) \approx \frac{b}{\omega_n^2} + C_1 e^{\zeta_1 t} \]

This has the same form as a first-order step response with time constant \( \zeta_1 \).
Ex) \[ \ddot{y} + 11 \dot{y} + 10y = 10u(t) \]

\[
\omega_n^2 = 10 \quad \Rightarrow \quad \omega_n = \sqrt{10} \\
2\pi \omega_n = 11 \quad \Rightarrow \quad \phi = \frac{1}{2} \omega_n = \frac{1}{2} \sqrt{10} > 1 \quad \text{(system is overdamped)}
\]

The roots of the characteristic equation are at:
\[ r = -\phi \omega_n \pm \omega_n \sqrt{\phi^2 - 1} = -1, -10 \]

The step response solution has the form:
\[ y(t) = 1 + c_1 e^{-t} + c_2 e^{-10t} \]

Solve for \( c_1 \) and \( c_2 \) using the initial conditions:
\[ y(0) = 0 = 1 + c_1 + c_2 \quad \Rightarrow \quad c_1 = -10/9 \]
\[ y(0) = 0 = -c_1 - 10c_2 \quad \Rightarrow \quad c_2 = 1/9 \]

\[ y(t) = 1 + \left(\frac{-10}{9}\right) e^{-t} + \left(\frac{1}{9}\right) e^{-10t} \]

This term will decay faster than \((\frac{-10}{9}) e^{-t}\). This term has a time constant of \(\sqrt{\frac{10}{9}}\) sec.

\[ \therefore y(t) \approx 1 - \frac{10}{9} e^{-t} \quad \text{(after about 3/10 sec)} \]

Compare this response with the step response of:
\[ x(t) = -x(0) \quad \text{for} \quad x(0) = 0 \]
\[ y(t) = f(t) - e^{-t} \]

This first-order response is very similar to the second order response.

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Conclusion: For an overdamped second order system, there are two real poles \( r_1 \) and \( r_2 \). Each of these contributes a term to the step response of the form \( c_1 e^{r_1 t} \).

If both poles are negative, then these exponentials will decay to zero.

If \( |r_2| > |r_1| \) then \( e^{r_2 t} \) will decay much faster than \( e^{r_1 t} \).

In this case the second-order response is similar to that of a first order system with time constant \( \frac{1}{\sqrt{r_1 r_2}} \).
Next consider the underdamped case ($p<1$).

The step response has the form

$$y(t) = \frac{b}{\omega_n^2} + e^{-\frac{p}{\omega_n} t} \left[ A \cos \left( \omega_n \sqrt{1-p^2} \ t \right) + B \sin \left( \omega_n \sqrt{1-p^2} \ t \right) \right]$$

Define $\omega_d = \omega_n \sqrt{1-p^2}$ = Damped Natural Frequency

$$y(t) = \frac{b}{\omega_n^2} + e^{-\frac{p}{\omega_n} t} \left[ A \cos (\omega_d t) + B \sin (\omega_d t) \right]$$

**Question:** How does the response depend on $p$?

Assume $\omega_n = 1$

As $p \rightarrow 0$, the response appears to be faster (it initially "rises" faster toward $b/\omega_n^2$) but it becomes more oscillatory. (The response overshoots past $b/\omega_n^2$ by a greater amount).

When $p=0$, the roots of the characteristic equation are on the imaginary axis and the system is marginally stable.