In this section we consider first-order linear ODEs:

\[
\begin{align*}
(1) \quad &x' + ax = bu \\
\qquad &x(0) = x_0
\end{align*}
\]

where \(a, b \in \mathbb{R}\) are constants and \(x, u \in \mathbb{R}\) is a scalar (1-d) signal. We'll provide the solution to this ODE for the following cases:

A) General Solution: Applicable for arbitrary inputs

B) Free (Initial Condition) Response: \(u(t) = 0 \quad \forall t \geq 0\)

C) Step Response: \(u(t) = \\begin{cases} 0 & t < 0 \\ u_0 & t \geq 0 \end{cases}\)

D) Impulse Response

E) Sineoidal Response

The responses for these classes of inputs will be derived later in the course.

In presenting the solution of (1) for inputs of type A) - C) we will introduce the following concepts

A) Stability

B) Time Constant

C) Steady-State Value

All of these results and concepts will be generalized to higher-order ODEs later in the course.
General Solution

Given the initial condition \( x(0) = x_0 \) and the arbitrary input function \( u(t) \) defined on \( t \geq 0 \), the solution of the first-order ODE \((1)\) is:

\[
(2) \quad x(t) = e^{-at} x_0 + \int_0^t e^{-a(t-s)} u(s) \, ds
\]

\( \text{Free Response} \) \hspace{2cm} \text{Forced Response}

\( \text{for "Initial Condition" Response} \)

You can verify that \((2)\) is the solution to \((1)\) by checking that:

1. \( x(t) \) satisfies the given initial condition.
2. \( x(t) \) satisfies the ODE \((1)\).

It can also be shown that this solution is unique.

We'll derive this solution using the integrating factor method. Recall the chain rule for differentiation:

\[
\frac{d}{dt} \left[ F(t) x(t) \right] = F(t) \frac{d}{dt} x(t) + f(t) x(t)
\]

The method tries to multiply both sides of the ODE \((1)\) by \( F(t) \) so that \( F(t)x(t) \) can be expressed as

\[
\frac{d}{dt} \left[ F(t) x(t) \right]
\]

using the chain rule. \( F(t) \) is the integrating factor.
For the first-order ODE, choose \( f(t) = e^{at} \) and multiply both sides of the ODE by this factor:

\[
e^{at} \left[ x(t^2 + ax) \right] = e^{at} \left[ b u(t) \right]
\]

Using the chain rule, the left-side can be simplified to:

\[(x3) \quad \frac{d}{dt} \left[ e^{at} x(t) \right] = e^{at} b u(t)\]

The key point is that the left side is now an exact differential and can be integrated. The choice of the integrating factor \( f(t) = e^{at} \) appears here as a guess. However, with some thought it can be seen that \( f(t) \) must have this form in order to apply the chain rule.

Integrating both sides of (x3) yields:

\[
\int_0^t e^{az} b u(z) \, dz = \int_0^t \frac{d}{dz} \left[ e^{az} x(z) \right] \, dz
\]

\[
= \left[ e^{az} x(z) \right]_0^t = e^{at} x(t) - x(0)
\]

\[
\Rightarrow e^{at} x(t) - x(0) = \int_0^t e^{az} b u(z) \, dz
\]

\[
\Rightarrow \quad x(t) = e^{-at} x(0) + \int_0^t e^{-a(t-z)} b u(z) \, dz
\]

This is the general solution to the first-order model (x1) for arbitrary inputs. This formula can be generalized to state-space models with \( n \)-states.
Ex) RC Circuit

Applying Kirchoff's Voltage Law:
\[ V_C(t) + \frac{1}{RC} V_C(t) = \frac{1}{RC} V_i(t) \]

Applying the general solution (\(x2\)), the output voltage for an arbitrary initial condition \(V_C(0)\) and input voltage \(V_i(t)\) is given by:
\[ V_C(t) = e^{-t/RC} V_C(0) + \int_0^t e^{-(t-z)/RC} \frac{V_i(z)}{RC} \, dz \]

Suppose \(V_i(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases} \) This is a unit-step voltage at \(t=0\).

The solution is:
\[ V_C(t) = e^{-t/RC} V_C(0) + \int_0^t e^{-(t-z)/RC} \frac{1}{RC} \, dz \]
\[ = e^{-t/RC} V_C(0) + e^{-t/RC} \frac{1}{RC} \left[ e^{t/RC} - 1 \right] \]
\[ = e^{-t/RC} V_C(0) \left( 1 - \frac{1 - e^{-t/RC}}{RC} \right) \]

If \(V_C(0) = 0\), then the response is:
\[ V_C(t) = 1 - e^{-t/RC} \]

\[ \approx 0.95 \]

\[ 1 \]

\[ 3RC \]

\[ V_C(t) \]
The free response, also called the initial condition response, is the solution for the case $u(t) = 0$ for $t > 0$. The solution of the first-order ODE (1) for $u(t) = 0$ for $t > 0$ is given by:

$$x(t) = e^{-at}x_0$$

There are many ways to derive this solution. A simple way is to plug $u(t) = 0$ into the previously derived general solution (2). The characteristics of this solution depend on the value of $a$.

If $a > 0$:

Note the following important characteristics of the solution:

1) $x(0) = -a x_0$. Thus the slope of the response at $t = 0$ is $-ax_0$. $T = \frac{1}{a}$ is the time that the initial slope crosses the horizontal axis if extended.

2) $x(3T) = e^{-a(3T)}x_0 = e^{-3}x_0 \approx 0.05x_0$.

Thus the free response (assuming $a > 0$) roughly decays to zero after $3$ time constants. Precisely, the response decays to about 0.5% of its original value after $3T$.

The time constant $T = \frac{1}{a}$ is a measure of the speed of response.
If \( a \leq 0 \) the free response does not decay back to \( 0 \).

\[ x(t) \]

\[ a < 0 \Rightarrow x(t) \to \infty \text{ as } t \to \infty \]

\[ a = 0 \Rightarrow x(t) = x_0 \text{ } \forall t \geq 0 \]

\[ a > 0 \Rightarrow x(t) \to 0 \text{ as } t \to 0 \]

Thus the response has 3 important cases:

- \( a > 0 \): The free response is stable.
- \( a = 0 \): The free response is marginally stable.
- \( a < 0 \): The free response is unstable.
Step Response

Again recall the first-order ODE (x1)
\[
\dot{x} + ax = bu
\]
\[x(0) = x_0\]

The step response is the solution of this ODE for the input \( u(t) = \begin{cases} 0 & t < 0 \\ u_0 & t \geq 0 \end{cases} \)

where \( u_0 \) is a constant. There are many ways to solve this ODE (Laplace transforms, particular/homogeneous solutions, etc.). We will again use the general solution (x2):
\[
x(t) = e^{-at}x_0 + \int_0^t e^{-a(t-s)}bu(s)\,ds
\]

Substituting the step input \( u(t) \) and performing the necessary integrations yields the solution:

If \( a \neq 0 \)
\[
x(t) = e^{-at}x_0 + \frac{b}{a} \left(1 - e^{-at}\right)u_0
\]

If \( a = 0 \)
\[
x(t) = e^{0t}x_0 + bu_0 \cdot t
\]

If \( a > 0 \) then the solution \( \frac{b}{a}u_0 = x_{ss} \)
\[
x_0 + 0.95(x_{ss} - x_0)
\]

Note:

1) If \( a > 0 \), \( x(t) \) eventually converges to \( x_{ss} = \frac{b}{a}u_0 \). This is called the steady-state. The steady-state is actually an equilibrium point because \((x,\dot{x}) = (\frac{b}{a}u_0, x_0)\) satisfies \(\dot{x} = -ax + bu = 0\), i.e., the time derivative is zero at the steady-state.

2) The solution goes roughly 95% of the way from \( x_0 \) to \( x_{ss} \) after 3 time constants. Again, \( T = \frac{1}{a} \) is a measure of the speed of response.
If $a \leq 0$, then $x(t)$ does not converge to a steady-state.

We previously defined stability for a first-order system based on the free response. There is another notion of stability:

A system is **bounded-input, bounded-output (BIBO) stable** if the response stays bounded for all bounded inputs, i.e. if $u(t)$ is bounded (there is some $N < \infty$ such that $|u(t)| \leq N \forall t \geq 0$) then there is some $M < \infty$ such that $|x(t)| \leq M$ for all $t \geq 0$.

The first-order system is not BIBO stable if $a < 0$ because $x(t) \rightarrow \infty$ for a step input. The system is BIBO stable if $a > 0$ because we can show the response stays bounded for all bounded inputs (to be shown later in the course).

For linear systems, the two notions of stability are equivalent:

The free response decays to zero for any initial condition if and only if

The forced response remains bounded for all bounded inputs.
Ex) Sketch the step response of:

\[ 2x + 3x = 7u \]
\[ x(0) = 6 \]
\[ u(t) = 4 \quad \text{for} \quad t \geq 0 \]

First divide both sides by 2 to normalize the lead coefficient:

\[ \dot{x} + \frac{3}{2} x = \frac{7}{2} u \]

This is in the form of our standard first-order \((1,1)\) system with \(a = \frac{3}{2}, \ b = \frac{7}{2}\). Using the step response solution on p.7, we obtain

\[ x(t) = e^{-\frac{3}{2}t} \cdot 6 + \frac{7}{3} \left( 1 - e^{-\frac{3}{2}t} \right) \cdot 4 \]

However, we can quickly draw the step response even without relying on this explicit solution. In particular, the step step response must satisfy

1) \( x(0) = 6 \)
2) \( x(t) \to x_{ss} \) where the steady-state satisfies
   \[ \dot{x} + \frac{3}{2} x_{ss} = \frac{7}{2} u \Rightarrow \frac{3}{2} x_{ss} = \frac{7}{2} (4) \Rightarrow x_{ss} = \frac{28}{3} \approx 9.33 \]
   \[ x_{ss} = \frac{28}{3} \text{in ss} \]
3) The response is exponential from \( x(0) \) to \( x_{ss} \) with
4) time constant \( T = \frac{1}{a} = \frac{2}{3} \). It takes roughly
   \[ 3T \text{ to go from } x(0) \text{ to } x_{ss} \]
   \[ 95\% \] of the way from \( x(0) \) to \( x_{ss} \),

\[ \frac{28}{3} = x_{ss} \]
\[ 6 = x(0) \]

\[ 3T = 2 \text{ seconds} \]
Summary of First-order Response

We've been studying first-order linear ODEs of the form:
\[ x + a \cdot x = b u, \quad x(0) = x_0 \]

The key points are:

1) Given \( x(0) = x_0 \) and \( u(t) \) defined on \( t > 0 \), the general solution is:
\[ x(t) = e^{-at} x_0 + \int_0^t e^{-a(t-\tau)} b u(\tau) d\tau \]

2) The speed of response (decay rate) is governed by the time constant:
\[ T = \frac{1}{a} \]

3) The free (initial condition) response is \( x(t) = e^{-at} x_0 \).
   
   3a) The system is stable (free response decays to zero for any \( x(0) \)) if \( a > 0 \). In this case, the response decays to 0.05\( x_0 \) after 3 time constants.
   
   3b) The system is unstable (free response grows exponentially) if \( a < 0 \).
   
   3c) The system is marginally stable if \( a = 0 \).

4) The response due to a step input \( u(t) = u_0 \) \( t > 0 \) is:
   \[ x(t) = e^{-at} x_0 + \frac{b}{a} (1 - e^{-at}) u_0 \quad \text{(for } a > 0) \]
   \[ x(t) = x_0 + b u_0 \cdot t \quad \text{(for } a = 0) \]

4a) If \( a > 0 \) then the response converges to a steady-state (constant) value \( x_{ss} = \frac{b}{a} u_0 \). It takes roughly 3T to go 95% of the way from \( x_0 \) to \( x_{ss} \).

4b) If \( a < 0 \) then \( x(t) \to \infty \) as \( t \to \infty \).