Robustness Margins

We can use the Nyquist stability condition to develop more general robustness conditions. Recall that we defined gain and phase margins as:

a) Gain Margin: The amount of variation in the plant gain that can be allowed before the open-loop goes unstable.

\[ G_c(s) = e^{\gamma} G(s) \quad \text{models the gain variation with } \gamma = 1 \text{ representing the nominal dynamics.} \]

\[ L_c(s) = G_c(s) \cdot K(s) = e^{\gamma} [G(s) \cdot K(s)] = e^{\gamma} L(s) \]

We showed how to use the Bode plot to compute \( \gamma \) and \( \overline{\gamma} \) such that the closed-loop remains stable for all \( \gamma \) in \( \gamma < \gamma < \overline{\gamma} \) [assuming the nominal system for \( \gamma = 1 \) is stable].

b) Phase Margin: The amount of phase variation that can be allowed before the closed-loop goes unstable.

\[ G_\theta(s) = e^{\jmath \theta} G(s) \quad \text{models the phase variation with } \theta = 0 \text{ representing the nominal dynamics.} \]

\[ L_\theta(s) = e^{\jmath \theta} L(s) \]

We showed how to compute \( \overline{\theta} \) such that the closed-loop remains stable for all \( \theta \) in \( -\overline{\theta} < \theta < \overline{\theta} \) [Again, assuming the nominal system for \( \theta = 0 \) is stable].
Gain and phase margins can be measured on the Nyquist plot.

Ex) \( L(s) = \frac{9s}{s-1} \)

- \( L(s) \) has one pole in the RHP, thus its Nyquist plot must encircle \(-1\) once in the ccw direction to ensure closed-loop stability. The Nyquist plot (left) shows that it does encircle \(-1\) once in the ccw...
  
  \[ \text{closed-loop is stable.} \]

**Question** How much can \( c \) be varied and still have the closed loop with \( L_c(s) = \frac{9c}{s-1} \) be stable?

We can choose any \( c > 1 \) and the Nyquist plot of \( L_c(s) \) will still encircle \(-1\) the correct # of times. \( \Rightarrow \) Closed-loop is stable for all \( c > 1 \).

However, if we choose \( c \leq \frac{1}{9} \) then the Nyquist plot of \( L_c(s) \) will be shrunk down and it will no longer encircle \(-1\).

\( \Rightarrow \) Closed-loop is not stable for \( c \leq \frac{1}{9} \)

Thus, the gain margins are \( \delta = \frac{1}{9} \) and \( \epsilon = \infty \).

You should be able to draw the same conclusions from the Bode plot of \( L(s) \).
The margins (gain and phase) can be computed using either half of the Nyquist plot (either the part due to $s=j\omega$ with $\omega>0$ or the part due to $\omega<0$). Focus on an example Nyquist plot shown below only for $L(j\omega)$ with $\omega>0$:

\[ L(j\omega) = -0.4 \]

Unit Circle (Circle of Radius 1)

\[ L(j\omega_1) = -3.3 \]
\[ \angle L(j\omega_2) = -135^\circ \]
\[ |L(j\omega_2)| = 1 \]

A) Gain Margin: Recall that we computed the gain margins by finding all phase cross-over frequencies, i.e. frequencies where $\angle L(j\omega) = -180^\circ$. These correspond to frequencies where $L(j\omega)$ is a real, negative number. On the Nyquist plot above, frequencies $\omega_1$ and $\omega_3$ are phase cross-over frequencies. We then computed the critical gains as $\frac{1}{\sqrt{|L(j\omega)|}}$ at the phase cross-over frequencies. For the example above we get

\[ \omega_1 \rightarrow \frac{1}{\sqrt{|L(j\omega)|}} \approx 0.3 \Rightarrow \text{A gain of } c=0.3 \text{ will cause the Nyquist plot of } L_c = CL \text{ to pass through } -1 \text{ at } s=j\omega_1 \text{. The closed-loop will have a pole at } s=j\omega_1 \]

\[ \omega_2 \rightarrow \frac{1}{\sqrt{|L(j\omega)|}} = 2.5 \Rightarrow \text{A gain of } c=2.5 \text{ will cause the Nyquist plot of } L_c = CL \text{ to pass through } -1 \text{ at } s=j\omega_3 \text{. The closed-loop will have a pole at } s=j\omega_3 \]

The closed loop will be stable for $2.5 < c < 10$. 
B) Phase Margin: Recall that we computed the phase margins by finding all gain cross-over frequencies, i.e. frequencies where \(|L(\omega)\| = 1\). These correspond to frequencies where the Nyquist plot of \(L\) intersects the disk of radius 1. On the example on p.257, \(\omega_2\) is a gain cross-over frequency. We then compute the phase margin by the amount of phase (in absolute value) that causes \(e^{i\theta}L(s)\) to have \(180^\circ\) of phase. For the example above

\[
\omega_2 \Rightarrow |L(\omega_2)| = -130^\circ
\]

\[\Rightarrow \text{An additional } -50^\circ \text{ of phase would cause } e^{i\theta}L \text{ to have } -180^\circ \text{ of phase.}\]

\[\Rightarrow \text{The phase margin is } 50^\circ\]

\[\Rightarrow \text{The closed loop is stable for } -\bar{\theta} = -50^\circ < \bar{\theta} = +50^\circ = \bar{\theta}.
\]

Note that if we add \(\bar{\theta} = -50^\circ\) of phase to \(L(s)\) then the Nyquist plot of \(e^{i\theta}L(s)\) will pass through \(-1\) at \(s = j\omega_2\). Thus the closed-loop will be unstable with a pole at \(s = j\omega_2\).
Robustness Margins measure how close the Nyquist plot of $L$ approaches to the $-1$ point. It is assumed that the controller $K(s)$ has been designed so that $L(s) = G(s)K(s)$ encircles $-1$ the "correct" number of times to ensure closed-loop stability. However, if the Nyquist plot of $L$ comes "near" to $-1$ then small variations in the model $G(s)$ can cause $L(j\omega)$ to pass through $-1$ or $L(j\omega)$ to have the wrong number of encirclements of $-1$.

The classical gain and phase margins measure the distance from $L(j\omega)$ to $-1$ only along 2 specific directions:

- Gain margins measure the distance between $L(j\omega)$ and $-1$ along the negative imaginary axis.
- Phase margins measure the distance between $L(j\omega)$ and $-1$ along the circle of radius $1$.

These margins are a control engineer's version of "safety factors". As mentioned previously, $\pm 6$dB of gain margin and $\pm 45^\circ$ of phase margin are good rules of thumb for robustness.
Extensions to Gain and Phase Margins

Gain and phase margins are classical robustness margins that can be read from the Bode plot of the open loop transfer function. As discussed on the previous page, these classical margins measure, in some limited way, how close $L(j\omega)$ comes to -1.

There are many extensions to the classical margins for measuring the robustness of a feedback system. The minimum distance of the Nyquist plot $L(j\omega)$ to -1 is a good robustness measure.

This distance is:

$$\min_{\omega} \left| \frac{1}{1+L(j\omega)} \right|$$

If this distance is small then small variations in the plant model can cause $L(j\omega)$ to pass through -1 or cause $L(j\omega)$ to have the "wrong" number of encirclements. In both cases this would cause the closed loop to be unstable.

We can connect this metric to the sensitivity function $S(s) = \frac{1}{1+L(s)}$. Specifically,

$$\left| \frac{1}{1+L(j\omega)} \right| < d \quad \text{for all } \omega$$

if and only if

$$|S(j\omega)| = \frac{1}{1+L(j\omega)} \leq \frac{1}{d} \quad \text{for all } \omega$$

Thus large peak values of $|S(j\omega)|$ correspond to poor robustness ($L(j\omega)$ comes near -1) and vice versa; good robustness is ensured by small peak values of $|S(j\omega)|$ ($|L(j\omega)|$ stays away from -1).
As a rule of thumb, \(|S(j\omega)| \leq 2.5\) for all \(\omega\) will provide good robustness margins.

This is equivalent to \(|1 + L(j\omega)| \geq \frac{1}{2.5} = 0.4\) for all \(\omega\).

This means that the Nyquist plot \(L(j\omega)\) has a minimum distance of at least 0.4 from -1.

Graphically, \(L(j\omega)\) does not enter the disk of radius 0.4 centered at -1.

If \(|S(j\omega)| \leq 2.5\) for all \(\omega\), then \(L(j\omega)\) does not enter the shaded disk. This ensures that \(L(j\omega)\) has the at least the following classical margins:

1) \(\bar{C} = 10 \text{ Upper Gain Margin} \geq \frac{1}{0.16} = 1.67\) dB

2) \(\underline{C} = 10 \text{ Lower Gain Margin} \leq \frac{1}{0.4} = 0.71\)

3) \(\bar{\Theta} = \text{Phase Margin} \geq 37^\circ\)

1-3 are slightly less than our previous rules of thumb of \(\pm 6\text{dB} (\bar{C} \geq 2, \underline{C} \leq 1\text{dB})\) and \(\pm 45^\circ\). However, 1-3 are minimum classical margins ensured by \(|S(j\omega)| \leq 2.5\) and the actual values will typically be better.
Basic Loopshaping Theorem

Our basic loopshaping design procedure is based on 3 characteristics:

a) Make \( |L| \) large at low frequencies so that \( |S| \) is small, i.e., good tracking at low frequencies.

b) Make \( |L| \) small at low frequencies so that \( |T| \) is small, i.e., good noise rejection at high frequencies.

c) Make the slope of \( |L| \) "shallow" \( (-30 \text{ dB/dec}) \) at the gain crossover (mid frequencies) so that the closed-loop is stable and has good robustness.

It was fairly straightforward to make the connection between the loop gain \( |L| \) and the requirements on \( |S| \) and \( |T| \) (characteristics a and b). For example, see lecture 31.

We also used the Bode gain/phase formula to show that if the slope of \( |L| \) is \(-30 \text{ dB/dec}\) for approximately 1 decade of frequency surrounding the gain crossover then the system will have approximately \( \pm 45^\circ \) of phase margin. For example see p. 212-215. This is part of characteristic c but we still don't have any assurances that the closed-loop will actually be stable. We'll use the Nyquist stability theorem to show that our loop-shaping design will achieve a stable closed-loop.
Theorem

Assume the loop transfer function \( L(s) \) satisfies:

- \( L(s) \) has no poles or zeros in the closed RHP
- \( L(0) > 0 \)
- \( L \) has one gain crossover frequency \( \omega_g \)
- The slope of \( |L| \) is \(-30 \text{ dB/dec}\) for at least 1 decade of frequencies around \( \omega_g \): \( \omega_e \leq \omega_g \leq \omega_h \)
- Outside this frequency interval around \( \omega_g \):
  - \( |L| \geq 2 \) at lower frequencies \( \omega \leq \omega_e \)
  - \( |L| \leq 1 \) at higher frequencies \( \omega \geq \omega_h \)

If all these conditions are satisfied, then the closed-loop is stable, approximately achieving good classical margins \((6 \text{ dB}, 150^\circ)\) and has \( |\Delta(j\omega)| \leq 2.5 \) for all \( \omega \).

Before we sketch a proof of this result, there are a few important remarks:

a) The theorem can be extended/generalized to systems with poles and/or zeros in the closed RHP. An important case is loop transfer functions with integrators (poles at \( s = 0 \)).

b) The condition \( L(0) > 0 \) is easy to satisfy. If \( G_a > 0 \) then choose \( K(0) > 0 \). If \( G_a < 0 \) then choose \( K(0) < 0 \).
Proof (Due to A. Packard)

The proof uses the Bode gain/phase relation and the Nyquist stability theorem. We'll sketch the constraints on the Nyquist plot of \( L \) based on the given assumptions.

\( L \) has one gain crossover frequency where \( |L(j\omega_{gy})| = 1 \). Since the slope of \( \|L\| > -30 \text{dB/dec} \) for at least one decade around \( \omega_{gy} \), the accumulated phase must approximately satisfy
\[
\angle L(j\omega_{gy}) > -135^\circ \quad \text{by the Bode gain/phase formula}.
\]
For lower frequencies (\( \omega < \omega_{gy} \)) the Nyquist plot lies outside the disk of radius 2 and for higher frequencies (\( \omega > \omega_{gy} \)) the Nyquist plot lies inside a disk of radius \( \frac{1}{2} \). Thus it is not possible for the Nyquist plot of \( L \) to encircle -1. By the Nyquist theorem:
\[
\text{(\# of closed-loop)} - \text{(\# of open loop)} - \text{(\# of CW encirclements of -1)} = 0
\]
Thus the closed-loop will be stable. Moreover, the system will appear have ±45° of phase margin. In addition, any phase crossover frequencies, \( \angle L(j\omega) = -180^\circ \), can only occur when \( \|L\| > 2 \) or < \( \frac{1}{2} \). Thus the system will appear have ±6dB of gain margin. Finally, \( L(j\omega) \) will not enter the disk of radius 0.4 centered at -1. Therefore \( 1.5 < |Z| \) at all \( \omega \) because \( 1+L \approx 0.4 \) at all...