Equilibrium Points

Consider a nonlinear differential equation

\[ \dot{x} = f(x, u) \quad (\ast) \]

where \( x \in \mathbb{R}^n \) and \( u \in \mathbb{R}^m \). The notation \( \mathbb{R}^n \) denotes \( n \)-dimensional vectors of real numbers.

A point \( \bar{x} \in \mathbb{R}^n \) is called an equilibrium point if there is an input \( \bar{u} \in \mathbb{R}^m \) (called the equilibrium input) such that \( f(\bar{x}, \bar{u}) = 0 \). This is called an equilibrium point because if the system is initialized to \( x(0) = \bar{x} \) and the input is held constant at \( u(t) = \bar{u} \) for all \( t \geq 0 \), then the solution of \( (\ast) \) satisfies \( x(t) = \bar{x} \) \( \forall t \geq 0 \).

Finding \( (\bar{x}, \bar{u}) \) such that \( f(\bar{x}, \bar{u}) = 0 \) is often called "trimming" the system. This requires solving \( n \) equations for \( n+m \) unknowns. The equilibrium point is not unique since there are fewer equations than unknowns.
Recall the 2nd order ODE model (nonlinear) for the pendulum:
\[ \ddot{\theta} + \frac{g}{l} \sin \theta = \frac{1}{l} \dot{\gamma} \]

We previously converted this to the following nonlinear state-space model:
\[ \begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{g}{l} \sin(x_1) + \frac{1}{m l^2} u
\end{align*} \]

This is in the standard form
\[ \dot{x} = f(x, u) \]

where \( x = [x_1, x_2] \) and \( f(x, u) = [-\frac{g}{l} \sin(x_1) + \frac{1}{m l^2} u] \)

First consider the case \( u(t) = 0 \) \( \forall t \geq 0 \), i.e. there is no external input torque on the pendulum. An equilibrium point requires
\[ 0 = \dot{x} = f(x, 0) \]

\[ \begin{cases} 
0 = -\frac{g}{l} \sin(x_1) \\
0 = \frac{1}{m l^2} u
\end{cases} \]

Mathematically there are an infinite number of equilibrium points. However there are only 2 physical equilibrium points
A) \( (\bar{x}, \bar{u}) = (0, 0) \)
B) \( (\bar{x}, \bar{u}) = \left( \frac{\pi}{2}, 0 \right) \)

The remaining mathematical eq. pts. \( (\left[ \begin{array}{c} k \pi \\pi \end{array} \right], 0) \) for some integer \( k \) corresponds to rotations of the pendulum relative to either A) or B).
A) \((\bar{X}, \bar{\theta}) = ([0], 0)\)

Physically this eq. pt. corresponds to \(\theta = 0\) (pendulum is not rotating) and \(\theta = 0\) (pendulum in the downward position). \(\bar{\theta} = 0\) again means there is no input torque.

If the pendulum is initially at \(X = \bar{X}\) and the input is held at zero \(u(t) = \bar{u}\) \(\forall t \geq 0\) then the pendulum will remain in the downward position forever \(X(t) = \bar{X} = [0] \quad \forall t \geq 0\). This is the basic meaning of an equilibrium point.

B) \((\bar{X}, \bar{\theta}) = ([\pi], 0)\)

Physically this eq. pt. corresponds to \(\theta = 0\) (pendulum is not rotating) and \(\theta = \pi\) (pendulum is in the upward position). Again, if the pendulum is initialized at this position then it will stay in this position forever. This assumes that there are absolutely no disturbances acting on the pendulum and the pendulum starts exactly in the upward direction. These are not very realistic assumptions but the concept of an eq. pt. is still useful.
Next consider the case where \( u(t) \) can be non-zero. Again an eq. pt. and eq. input require

\[
0 = \dot{x} = F(x, u)
\]

\[
\Rightarrow \begin{cases} 
0 = x_2 \\
0 = -\frac{g}{l} \sin(x_1) + \frac{1}{m} \ddot{x} u \\
\end{cases} \Rightarrow \begin{cases} 
x_2 = 0 \\
u = mg l \sin(x_1) \\
\end{cases}
\]

There is a whole family of equilibrium points:

\[
(x, u) = \left( \begin{bmatrix} x_1 \\ 0 \end{bmatrix}, \frac{mg l \sin(x_1)}{m} \right)
\]

We can make the pendulum stay at any angle \( x_1 \) by choosing the input torque as \( u = mg l \sin(x_1) \). Specifically, if the pendulum is initialized to

\[
x(0) = x = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}
\]

and we hold the input torque as \( u(t) = mg l \sin(x_1) \) \( \forall t \geq 0 \) then the pendulum will stay at the angle \( x_1 \) (\( x(t) = \begin{bmatrix} \tilde{x} \\ 0 \end{bmatrix}, \forall t \geq 0 \)).
If a nonlinear system operates "near" an equilibrium point then the dynamics can be approximated by a linear ODE. The precise steps of this approximation are known as Jacobian Linearization. This is extremely useful because most of tools are applicable for linear systems.

Before describing the linearization method for nonlinear systems we first recall the Taylor Series expansion. You likely encountered encountered the Taylor Series expansion in your Calculus courses.

Let $F: \mathbb{R} \to \mathbb{R}$. This notation means $F$ takes $x \in \mathbb{R}$ as an input and returns $F(x) \in \mathbb{R}$. The Taylor series around a point $\bar{x}$ is

$$F(x) = F(\bar{x}) + \frac{dF}{dx}(\bar{x}) \cdot (x - \bar{x}) + \text{Higher Order Terms}$$

If $x$ is "near" $\bar{x}$ then

$$F(x) \approx F(\bar{x}) + \frac{dF}{dx}(\bar{x}) \cdot (x - \bar{x})$$

The error in making this approximation is on the order of $(x - \bar{x})^2$.

Ex) Consider $F(x) = x^2$ near $\bar{x} = 3$.

The Taylor series is:

$$F(x) = F(3) + \frac{dF}{dx}(3) \cdot (x - 3)$$

$$\Rightarrow F(x) \approx 9 + 6(x - 3)$$

If $x$ is near $\bar{x} = 3$ then $F(x)$ is approximately equal to $9 + 6(x - 3)$.

If $x = 3.1$ then $F(3.1) = 9.61$ and the linearization is $9 + 6(3.1 - 3) = 9.6$.
We can use the Taylor series expansion to approximate the dynamics of a nonlinear system near an equilibrium point. First consider a 1-state nonlinear system (we'll present the general case later):

\[ \dot{x} = f(x, u) \]  \hspace{1cm} (x1)

where \( x \in \mathbb{R} \), \( u \in \mathbb{R} \), and \( f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \). The notation \( f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) means \( f \) takes 2 inputs \((x, u)\) which are both real numbers and returns a real number \( f(x, u) \).

Assume \((\bar{x}, \bar{u})\) is an equilibrium point: \( f(\bar{x}, \bar{u}) = 0 \).
We know that if the system starts at \( x(0) = \bar{x} \) and we hold the input at \( u(t) = \bar{u} \) \( \forall t \geq 0 \) then the solution of (x1) stays at \( x(t) = \bar{x} \) \( \forall t \geq 0 \).

**Question:** What happens if \( x(0) \) is slightly different from \( \bar{x} \) and/or the input \( u(0) \) is slightly different from \( \bar{u} \)?

We could answer this question by directly solving (x1). However, many of our design and analysis tools are developed for linear ODEs. Thus it would be useful to derive a linear ODE that provides an approximate solution to the nonlinear ODE (x1).

We can perform a multi-variable Taylor series expansion around \((\bar{x}, \bar{u})\) to approximate the nonlinear function in (x1):

\[ f(x, u) \approx f(\bar{x}, \bar{u}) + \frac{\partial f}{\partial x}(\bar{x}, \bar{u})(x - \bar{x}) + \frac{\partial f}{\partial u}(\bar{u}, \bar{u})(u - \bar{u}) \]  \hspace{1cm} (x2)

We have dropped higher-order (quadratic, etc.) terms. In addition, we have generalized the single-variable Taylor series used on p5 because \( f(x, u) \) has 2 input variables.
Now \( f(x, u) = 0 \) because \((x, u)\) is an equilibrium point. Thus if we substitute \((x, u)\) into \((x1)\) we obtain:

\[
(x3) \quad \dot{x}(t) = \left[ \frac{\partial f}{\partial x}(x, u) \right] \cdot (x - \bar{x}) + \left[ \frac{\partial f}{\partial u}(x, u) \right] \cdot (u - \bar{u})
\]

Finally, we introduce/define a few new variables so that \((x3)\) can be put in a more standard form:

\[
A = \frac{\partial f}{\partial x}(x, u) \quad \text{These are just a constant real #}
\]

\[
B = \frac{\partial f}{\partial u}(x, u)
\]

\[
\delta_x(t) = x(t) - \bar{x} \quad \text{These variables measure the deviation from the equilibrium point},
\]

\[
\delta_u(t) = u(t) - \bar{u}
\]

\[
dt \delta_x(t) = \delta_x(t) \quad \text{Note that } \delta_x(t) = \frac{d}{dt}(x(t) - \bar{x}) = \dot{x}(t) \quad \text{because } \bar{x} \text{ is a constant.}
\]

With this notation we can rewrite \((x3)\) as:

\[
(x4) \quad \frac{d}{dt} \delta_x(t) = A \delta_x(t) + B \delta_u(t)
\]

This is a linear state-space model. The solution of the linear model \((x4)\) will approximate the solution of the nonlinear model \((x1)\) as long as \(x(t)\) stays "near" \(\bar{x}\) and \(u(t)\) stays "near" \(\bar{u}\).

Specifically, let \(x(t)\) be the solution of the nonlinear system \((x1)\) starting from \(x(0)\) and with input \(u(t)\). Above let \(\delta_x(t)\) denote the solution of the linear system \((x4)\) starting from \(\delta_x(0) = x(0) - \bar{x}\) and with input \(\delta_u(t) = u(t) - \bar{u}\). Then the linear solution approximates the nonlinear solution (assuming small perturbations): \(x(t) = \bar{x} + \delta_x(t)\)
Ex 1: \[ \dot{x} = -3x^2 + 2\sin(x) + 4u \]

Let \( \bar{x} = 2 \) and \( \bar{u} = 3 - 0.5\sin(2) \approx 2.55 \). It can be checked that \( f(\bar{x}, \bar{u}) = 0 \) and hence \( (\bar{x}, \bar{u}) \) is an eq. pt.

Linearize the dynamics of the nonlinear ODE at \((\bar{x}, \bar{u})\):

\[
\dot{x} = f(x, u) \approx f(\bar{x}, \bar{u}) + \frac{\partial f}{\partial x}(\bar{x}, \bar{u}) \cdot [x - \bar{x}] + \frac{\partial f}{\partial u}(\bar{x}, \bar{u}) \cdot [u - \bar{u}]
\]

\[
\frac{\partial f}{\partial x}(\bar{x}, \bar{u}) = 4
\]

\[
\frac{\partial f}{\partial u}(\bar{x}, \bar{u}) = [\frac{\partial f}{\partial x}(\bar{x}, \bar{u})]_{(x, u) = (\bar{x}, \bar{u})} = -12 + 2\cos(2)
\]

Define \( A = -12 + 2\cos(2) \), \( B = 4 \), \( \delta x(t) = x(t) - \bar{x} \), and \( \delta u(t) = u(t) - \bar{u} \). The linear model at \((\bar{x}, \bar{u})\) is:

\[
\dot{\delta x}(t) = A\delta x(t) + B\delta u(t)
\]

The linear model provides a good approximation for the nonlinear model as long as \((x(t), u(t))\) stay near the eq. pt \((\bar{x}, \bar{u})\).

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The linear approx will be as good if \(x(0)\) is far from \(\bar{x}\) and/or \(u(0)\) is far from \(\bar{u}\).
Generalization

All of the steps on the previous pages can be generalized to obtain linearizations for \( n \)-state systems. The key concepts are exactly the same and the generalization mainly requires notational changes.

Consider the \( n \)-state nonlinear system:

\[
\dot{x} = f(x, u)
\]

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R} \) (single-input), and \( f: \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n \).

Assume \((\bar{x}, \bar{u})\) is an eq. pt., i.e. \( f(\bar{x}, \bar{u}) = 0 \).

The Taylor series expansion for the vector-valued function \( f(x, u) \) is given by (retaining only linear terms):

\[
f(x, u) \approx f(x, u) + \nabla_x f(\bar{x}, \bar{u}) \cdot (x - \bar{x}) + \nabla_u f(\bar{x}, \bar{u}) \cdot (u - \bar{u})
\]

where:

\[
\nabla_x f(\bar{x}, \bar{u}) = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n}
\end{bmatrix}
\]

\( \in \mathbb{R}^{n \times n} \) is an \( n \)-by-\( n \) real matrix.

\[
\nabla_u f(\bar{x}, \bar{u}) = \begin{bmatrix}
\frac{\partial f_1}{\partial u} \\
\vdots \\
\frac{\partial f_n}{\partial u}
\end{bmatrix}
\]

\( \in \mathbb{R}^n \) is an \( n \)-by-\( 1 \) real vector.
If we again define \( \delta x(t) = x(t) - \bar{x} \) and \( \delta u(t) = u(t) - \bar{u} \) then we obtain the linear state space model:

\[
\dot{\delta x}(t) = A \delta x(t) + B \delta u(t)
\]

This linear state-space model approximates the solution of the nonlinear state-space model.

Finally we note that nonlinear state-space models can also have an output equation to describe some measurement of interest:

\[
\begin{align*}
\delta x(t) &= f(\delta x, u) \\
y(t) &= h(\delta x, u) & \text{- output: Assume } y \in \mathbb{R}, \\
h: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}. 
\end{align*}
\]

The output can also be linearized. Briefly, if \( \delta y(t) = y(t) - \bar{y} \) where \( \bar{y} = h(\bar{\delta x}, \bar{u}) \) is the equilibrium output then

\[
\delta y(t) = \left[ \begin{array}{c} \nabla_x h(\bar{\delta x}, \bar{u}) \\ \nabla_u h(\bar{\delta x}, \bar{u}) \end{array} \right] \delta x(t) + \left[ \begin{array}{c} \nabla_y h(\bar{\delta x}, \bar{u}) \end{array} \right] \delta u(t)
\]

Combining this with the linear model above gives:

\[
\begin{align*}
\dot{\delta x}(t) &= A \delta x(t) + B \delta u(t) \\
\delta y(t) &= C \delta x(t) + D \delta u(t)
\end{align*}
\]

The same form of a state-space model form can be derived even if there are multiple inputs and multiple outputs.
Ex 1 Consider the nonlinear ODE:
\[
\begin{align*}
\dot{x}_1 &= x_2^2 - y x_1 \\
\dot{x}_2 &= -9 x_1 + u^2 \\
y &= x_1 x_2
\end{align*}
\]
where \( f(x,u) = \begin{bmatrix} x_2^2 - y x_1 \\ -9 x_1 + u^2 \end{bmatrix} \) and \( h(x,u) = x_1 x_2 \).

\((\bar{x},\bar{u}) = (1, 3)\) is an eq. pt as it can be verified that \( f(\bar{x},\bar{u}) = 0 \). The equilibrium output at this point is \( \bar{y} = h(\bar{x},\bar{u}) = 2 \).

Compute the Jacobian linearization at this eq. pt.

\[
\begin{bmatrix} -y & 2x_2 \\ -9 & 0 \end{bmatrix} \rightarrow \nabla_x f(\bar{x},\bar{u}) = \begin{bmatrix} -4 & 7 \end{bmatrix} = A
\]

\[
\begin{bmatrix} 0 & 2u \end{bmatrix} \rightarrow \nabla_u f(\bar{x},\bar{u}) = \begin{bmatrix} 0 \\ 6 \end{bmatrix} = B
\]

\[
\begin{bmatrix} x_2 & x_1 \end{bmatrix} \rightarrow \nabla_x h(\bar{x},\bar{u}) = \begin{bmatrix} 2 & 1 \end{bmatrix} = C
\]

\[
\nabla_u h(\bar{x},\bar{u}) = 0 \rightarrow \nabla_u h(\bar{x},\bar{u}) = 0 = D
\]

Define \( \delta x(t) = x(t) - \bar{x}, \delta u(t) = u(t) - \bar{u}, \delta y(t) = y(t) - \bar{y} \).

The state-space model (Jacobian linearization) for the nonlinear system around \((\bar{x},\bar{u})\) is:

\[
\begin{align*}
\dot{\delta}_x(t) &= A \delta x(t) + B \delta u(t) \\
\dot{\delta} y(t) &= C \delta x(t) + D \delta u(t)
\end{align*}
\]