Nyquist Stability Theorem

We can apply Cauchy's argument principle to derive a Nyquist's stability condition for feedback systems.

Consider the negative feedback system

If there are no pole/zero cancellations in the loop transfer function then the zeros of \( 1+L(s) \) are the closed-loop poles.

\[ \text{e.g., the transfer function from } r \rightarrow x \text{ is } \]
\[ T(s) = \frac{L(s)}{1 + L(s)}, \quad \therefore \text{zeros of } 1+L(s) \]
are poles of \( T(s) \).

Let \( L(s) = \frac{n(s)}{d(s)} \) where \( n, d \) are the numerator and denominator polynomials.

Nyquist Theorem

Assume \( L \) has no poles on the imaginary axis (we'll deal with open-loop imaginary poles soon).

Then,

\[ \left( \frac{\# \text{ of closed loop}}{\# \text{ of open loop}} \right) = \left( \frac{\text{poles in } \Gamma_R}{\text{poles in } \Gamma_R} \right) - \left( \frac{\# \text{ of times } L(\Gamma_R) \text{ encircles } -1 \text{ in the counterclockwise direction}}{} \right) \]

Moreover, if \( L(\Gamma_R) \) passes through -1 then the closed-loop system has a pole on \( \Gamma_R \).
Proof

Let \( L(s) = \frac{n(s)}{d(s)} \)

Define \( H(s) = 1 + L(s) = 1 + \frac{n(s)}{d(s)} \)

\[ H(s) = \frac{d(s) + n(s)}{d(s)} \]

Note that

1) Zeros of \( H(s) \) = Zeros of \( d(s) + n(s) \) = closed-loop poles

2) Poles of \( H(s) \) = Zeros of \( d(s) \) = open-loop poles.

Apply Cauchy's Argument principle with \( H(s) = 1 + L(s) \) and using the contour \( \Gamma_R \) (with sufficiently large \( R \)).

By Cauchy's argument principle,

\( H(\Gamma) \) will encircle zero \( N_z - N_p \) times where \( N_z \) = \# of zeros of \( H \) in \( \Gamma \)

\( N_p \) = \# of poles of \( H \) in \( \Gamma \)

\[ \text{(\# of clockwise encirclements)} = (\text{\# of closed-loop poles in} \ \Gamma) - (\text{\# of open loop poles in} \ \Gamma) \]
\[ d \left( \frac{\text{# of closed-loop poles in } \Gamma_R}{\text{poles in } \Gamma_R} \right) = \left( \frac{\text{# of open-loop poles in } \Gamma_R}{\text{poles in } \Gamma_R} \right) + \left( \frac{\text{# of CW encirclements of } 0 \text{ by } H}{\text{# of } H = 1+L} \right) \]

Since a negative # of CW encircling means it is equivalent to a positive # of CCW encircling, we can write the above relation as: where CW = clockwise, CCW = counterclockwise.

\[ \left( \frac{\text{# of closed-loop poles in } \Gamma_R}{\text{poles in } \Gamma_R} \right) = \left( \frac{\text{# of open-loop poles in } \Gamma_R}{\text{poles in } \Gamma_R} \right) - \left( \frac{\text{# of CCW encirclements of } 0 \text{ by } H}{\text{# of } -1 \text{ by } L} \right) \]

Finally, 1+L encircles the origin if and only if L encircles -1. Thus:

\[ \left( \frac{\text{# of closed-loop poles in } \Gamma_R}{\text{poles in } \Gamma_R} \right) = \left( \frac{\text{# of open-loop poles in } \Gamma_R}{\text{poles in } \Gamma_R} \right) - \left( \frac{\text{# of CCW encirclements of } -1 \text{ by } L}{\text{# of } H = 1+L} \right) \]

This proves the main statement of the Nyquist theorem.

If \( L(\Gamma_R) \) passes through -1 then \( L(s_0) = -1 \) for some \( s_0 \) on \( \Gamma_R \). \( s_0 \) is a pole of the closed-loop system.

It will typically occur that \( L \) will pass through -1 at some \( s_0 = \pm j\omega_0 \) on the imaginary axis segments of \( \Gamma_R \). In this case, the closed-loop system is marginally stable with poles at \( s = \pm j\omega_0 \).

**Remark:** Generally, we apply the Nyquist theorem with \( R \) sufficiently large (\( R \to \infty \)) such that \( \Gamma_R \) effectively contains the entire RHP. \( L(\Gamma_R) \) as \( R \to \infty \) is the Nyquist plot of \( L \).
Ex) \( L(s) = \frac{5}{s+1} \)

Nyquist plot \( L \)

\[ \text{[This is } L(R) \text{ as } R \to \infty] \]

By the Nyquist theorem,

\[
\text{\# of closed-loop poles in the RHP} = \left( \text{\# of open-loop poles in RHP} \right) - \left( \text{\# of times the Nyquist plot of } L \text{ encircles } -1 \right) = 0 \text{ because}
\]

\( L(s) = \frac{5}{s+1} \) has

no RHP poles

\[ \Rightarrow \text{\# of closed-loop poles in RHP} = 0 \]

Thus the negative feedback interconnection with \( L(s) = \frac{5}{s+1} \)
will be stable. This system is simple enough that we can directly verify this result:

\[
T(s) = \frac{L}{1+L} = \frac{\frac{5}{s+1}}{1+\frac{5}{s+1}} = \frac{5}{(s+1)s} = \frac{5}{5s+6}
\]

\[ \Rightarrow \text{The closed-loop has a pole at } s = -6 \text{ and hence is stable.} \]
\[ L(s) = \frac{5}{s-1} \]

For \( \omega > 0 \):
- \( \omega < 1 \Rightarrow L(j\omega) = -5 \)
- \( \omega > 1 \Rightarrow L(j\omega) \approx \frac{5}{j\omega} = \frac{-5}{\omega} \)
- \( \omega = 1 \Rightarrow L(j) = \frac{5}{j-1} = \frac{5}{2} (1-j) \)

Use this to draw the solid bottom half of the curve. The dashed top half is the curve for \( \omega < 0 \). It is the mirror image of the bottom half (i.e., \( L(-j\omega) \) is the complex conjugate of \( L(j\omega) \)).

By the Nyquist theorem,

\[
\text{# of closed loop poles in the RHP} = \left( \text{# of open loop poles in RHP} \right) - \left( \text{# of times the Nyquist plot of } L \text{ encircles } -1 \text{ in the ccw direction} \right) \]

= 1 because \( L(s) \) has one pole at \( s = 1 \) in the RHP

\[ \Rightarrow \text{# of closed loop poles in the RHP} = 0 \]  
(i.e., the closed-loop system is stable)

We can again directly verify this result since \( L \) is relatively simple:

\[ T(s) = \frac{L(s)}{1 + L(s)} = \frac{5}{1 + 5s - 1} = \frac{5}{(s-1)5} = \frac{5}{s-1} \]

\[ \Rightarrow \text{The closed-loop has a pole at } s = -4 \text{ and hence is stable.} \]
\[ L(s) = \frac{\frac{1}{2}}{s-1} \]

By the Nyquist theorem

\[
\text{\# of closed-loop poles in the RHP} = \left( \text{\# of open loop poles in RHP} \right) - \left( \text{\# of clockwise encirclements of -1 by the Nyquist plot of L} \right)
\]

\[
\text{\# of closed-loop poles in the RHP} = +1
\]

\[
\Rightarrow \text{The closed-loop system is unstable and has 1 RHP pole.}
\]

Verify this:

\[
T(s) = \frac{Y_1(s)}{1+L} = \frac{\frac{1}{2}}{1 + \left( \frac{1}{2} \right)} = \frac{1/2}{(s-1)+1} = \frac{1/2}{s-1/2}
\]

\[ T(s) \text{, the closed loop transfer function from } r \rightarrow y \text{ has one pole in the RHP at } s = \frac{1}{2}. \text{ It is unstable as predicted by the Nyquist theorem.} \]

Ex) \[ L(s) = \frac{50}{(s+1)^3} \]

This is a slightly more complicated example for you to try.

You should be able to:

a) Use Matlab's "nyquist" command to generate the Nyquist plot of \( L \).

b) Apply the Nyquist theorem to compute the \# of closed-loop poles in the RHP.

c) Verify your result by using the Matlab commands:

\[ T = \text{feedback}(L, 1); \quad p = \text{pole}(T) \]
A simplified version of the Nyquist theorem is:

**Nyquist Theorem (Simplified)**

The closed loop is stable if and only if the Nyquist plot of \( L \) encircles the -1 point the "correct" number of times.

I.e., a closed loop is stable if and only if the Nyquist plot of \( L \) encircles -1 in the counterclockwise direction the same number of times as the number of open-loop unstable poles.

**Example**

\[
L(s) = \frac{5}{s-1} \quad \frac{25s^2}{s^2 + 2(0.767)(25)s + 25^2}
\]

\( L \) has one pole in the RHP and the Nyquist plot of \( L \) encircles -1 once in the counterclockwise direction.

\[ \Rightarrow \text{closed-loop is stable.} \]