Nyquist Stability Theorem

We just introduced three simple metrics for the robustness of the closed-loop system: gain, phase, and time delay margins. In each case we assumed the closed-loop was stable and then we computed how much gain/phase/time delay variation could be tolerated before the system went unstable.

Next we'll discuss the Nyquist stability condition. This will serve several purposes including:

a) It will lead to a condition on the open loop transfer function $L(s)$ which is necessary and sufficient for closed-loop stability.

b) It will lead to more general robustness conditions than our simple gain/phase/time delay margins.

Nyquist Plot

The Nyquist plot is the basis for these more general stability results. Recall that a Bode plot of a system $G(s)$ consists of a gain plot, $\|G(jw)\|$ vs. $w$, and a phase plot, $\arg G(jw)$ vs. $w$.

A Nyquist plot is a single plot of $\text{Im}[G(jw)]$ vs. $\text{Re}[G(jw)]$. The convention is to draw this plot for both $w>0$ and $w<0$. The plot for $w<0$ is simply the mirror image of the plot for $w>0$:

$$G(-jw) = \overline{G(jw)} \Rightarrow \begin{cases} \text{Re}[G(jw)] &= \text{Re}[G(-jw)] \\ \text{Im}[G(jw)] &= -\text{Im}[G(-jw)] \end{cases}$$
Ex) \( G(s) = \frac{P}{s+p} \)

1. \( G(s) \) has a pole at \( s = -p \)
2. \( G(\omega) \approx 1 \) for \( \omega \ll p \)
3. \( G(\omega) \approx \frac{P}{j\omega} = \frac{P}{\omega} \) for \( \omega \gg p \)

In general,

\[
G(\omega) = \frac{P}{j\omega + p} = \frac{j\omega - p}{j\omega + p} = \frac{P^2 - j\omega P}{\omega^2 + p^2}
\]

\[
\text{Re}[G(\omega)] = \frac{P^2}{\omega^2 + p^2}
\]
\[
\text{Im}[G(\omega)] = \frac{-\omega P}{\omega^2 + p^2}
\]

\[
|G(\omega)| = \sqrt{\text{Re}[G(\omega)]^2 + \text{Im}[G(\omega)]^2} = \sqrt{\frac{P^4 + \omega^2 p^2}{\omega^2 + p^2}}
\]
\[
\angle G(\omega) = \text{arctan} \left( \frac{\text{Im}[G(\omega)]}{\text{Re}[G(\omega)]} \right) = \text{arctan} \left( -\frac{\omega}{p} \right)
\]

\( \uparrow \) Nyquist

\( \downarrow \) Bode

The dashed boxed half of the plot is \( G(\omega) \) for \( \omega \ll \omega_0 \). As mentioned before \( G(\omega) = \overline{G(j\omega)} \). Thus the curve for \( \omega \gg \omega_0 \) is the complex conjugate of the curve for \( \omega \ll \omega_0 \), i.e., it is a mirror reflection about the \( \text{Re}[G(j\omega)] \) axis.
\[ G(s) = \frac{9}{s-1} \]

- \( G(w) \) has a pole at \( s = 1 \)
- \( G(w) \approx -9 \) for \( w \ll 1 \) (\( |G(w)| \approx 9 \) and \( \angle G(w) \approx -180^\circ \))
- \( G(w) \approx \frac{9}{w} = -9j/w \) for \( w \gg 1 \) (\( |G(w)| \approx \frac{9}{w} \) and \( \angle G(w) \approx -90^\circ \))

**Bode**

\[ 20 \log |G| \]

**Nyquist**

\[ |m[G(w)]| \]

Again, the top half of the curve is \( G(jw) \) for \( w < 0 \). This is the mirror image of the bottom half which is \( G(jw) \) for \( w > 0 \).
To develop a Nyquist's stability theorem we need to briefly review one result from Complex analysis: Cauchy's Argument Principle. We'll sketch the basic idea, but more rigorous details can be found in Math 5583: Complex Analysis.

- Let \( G(s) \) be a rational function of \( s \).
- Let \( \Gamma \) be a simple, closed contour in the complex plane that does not pass through any poles or zeros of \( G(s) \).

[Simple = curve does not intersect itself]

**Convention:** \( \Gamma \) is traversed clockwise.

**G(s):**
- Map points \( s \) on \( \Gamma \) into their value \( G(s) \).

**G[\Gamma]:**
- Is the closed contour defined by mapping all of \( \Gamma \) by \( G \).

**Example:**
- \( G(s) = \frac{1}{s-1} \)
- If \( R \) is sufficiently large then \( G[\Gamma] \) is just the Nyquist plot of \( G(s) = \frac{1}{s-1} \) from the previous page.
Cauchy's Argument Principle:

Let $P = \# \text{ of poles of } G(s) \text{ inside of } \Gamma$

\[ P = \# \text{ of zeros of } G(s) \text{ inside of } \Gamma \]

Result: The number closed curve $G(\Gamma)$ encircles the origin $P = N_2 - N_1$ times, clockwise. If $N_2 - N_1 > 0$ then $G(\Gamma)$ encircles the origin clockwise. If $N_2 - N_1 < 0$ then $G(\Gamma)$ encircles the origin counterclockwise.

We won't prove this result but we'll give a few simple examples to explain the basic idea:

Ex) $G(s) = s - 1$

$\Gamma$: imaginary axis from $-j2$ up to $+j2$ and then the contour follows the circle of radius 2 back to $-j2$.

$G(\Gamma)$ has the same shape as $\Gamma$ but it is shifted to the left by 1 unit, e.g. $s = 0$ gets mapped to $G(0) = -1$.

$G(s)$ has one zero inside $\Gamma$ (at $s = 1$) and Cauchy's Argument principle states that $G(\Gamma)$ will encircle the origin once in the clockwise direction.
Ex) \( G(s) = s+1 \)

\[ G(s) \text{ has no poles or zeros inside } \Gamma' \text{ and hence Cauchy's argument principle states that } G(\Gamma') \text{ will encircle the origin 0 times.} \]

Ex) \( G(s) = (s-1)(s-2) \)

\[ G(s) \text{ has 2 zeros inside } \Gamma' \text{ (at } s=1 \text{ and } +2) \text{ and Cauchy's argument principle states that } G(\Gamma') \text{ will encircle the origin twice in the clockwise direction.} \]
Ex) \[ G(s) = \frac{9}{s-1} \]

\[ \Gamma = \text{imaginary axis from } -R \text{ to } R \text{ and then return along circle of radius } R. \]

The bottom half is the mapping of \( jw \) for \( 0 < w < R \) and the top half is the mapping of \( jw \) for \( -R < w < 0 \).

The small notch on the right is the mapping of the \( \text{or half-circle of radius } R. \) As \( R \to \infty \), \( G(\Gamma) \) converges to the N\( \text{yquist} \) plot of \( \frac{9}{s-1} \) (see p.228).

\[ G(s) \] has 1 pole inside \( \Gamma \) \( (s = 1) \) and Cauchy's argument principle states that \( G(\Gamma) \) will encircle the origin once in the counterclockwise direction \( (N_2 - N_p = 0 - 1 = -1) \).

\textbf{Note} \quad \text{If } \Gamma \text{ passes through a zero of } G \text{ then Cauchy's argument principle does not apply. In this case, if } \Gamma \text{ passes through a zero of } G \text{ then } G(\Gamma) \text{ will pass through 0. In the example above } G(s) \to 0 \text{ as } s \to \infty. \text{ Thus } \Gamma \text{ passes through a zero of } G \text{ as } R \to \infty \text{ and hence } G(\Gamma) \text{ passes through 0 as } R \to \infty. \]