Loopshaping Design Stages

Our basic design procedure is to "shape" the gain of the loop transfer function $L(s) = G(s) K(s)$. Specifically, we'll choose $K(s)$ so that $|L(j\omega)| = |G(j\omega) K(j\omega)|$ meets certain design specifications. One key point is that $K$ has an additive effect on the Bode magnitude plot:

$$20 \log_{10} |L(j\omega)| = 20 \log_{10} |G(j\omega) K(j\omega)|$$

$$= 20 \log_{10} |G(j\omega)| + 20 \log_{10} |K(j\omega)|$$

Thus our choice of $K$ simply adds to the magnitude of $G$ when we work in units of dB. This will make it easy to understand the impact of our design choices.

We'll use the following design "stages" as basic building blocks in shaping the loop:

1) Proportional Gain
2) Integral Boost
3) Low Frequency Boost Boost
4) Roll off
5) Lead
6) Lag

We'll describe each of these individual stages on the following pages. Our controller will end up being a product of these stages.
**Proportional Gain**

\[ K(s) = \beta \]  
where \( \beta \) is the gain.

The effect of the proportional gain is to shift the entire loop shape.

**Ex)** Plant, \( G: \ x = u \rightarrow G(s) = \frac{1}{s} \)

Proportional Gain: \( K(s) = \beta \)

Consider 2 cases:
1) \( K_1(s) = 10 \rightarrow L_1(s) = G(s)K_1(s) = 10\frac{1}{s} \)
2) \( K_2(s) = 0.1 \rightarrow L_2(s) = G(s)K_2(s) = 0.1\frac{1}{s} \)

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**Note:** The gain of \( |K_1| = 10 \) corresponds to a shift of +20dB  
and the gain of \( |K_2| = 0.1 \) corresponds to a shift of -20dB.
Integral Boost: Use to increase gain for $\omega < \omega$.

$$K(s) = \frac{s + \omega}{\omega}$$ where $\omega$ is the chosen corner frequency.

**Properties**

- $K(s)$ has a pole at $s = 0$ and a zero at $-\omega$.
- At low frequencies, $K(j\omega) \approx \frac{\omega}{j\omega}$.

For every factor of 10 decrease in frequency, $K(j\omega)$ increases in gain by a factor of 10.

[If $\omega_1 = \frac{\omega_2}{10}$ and $\omega_1, \omega_2 < \omega$ then $K(j\omega_1) \approx 10K(j\omega_2)$]

- At high frequencies, $K(j\omega) \approx 1$ [\(= 0 \text{ dB}\)]

- This transfer function corresponds to a PI controller with proportional gain $= 1$ and integral gain $= \omega$.

$$U(s) = K(s)E(s) \Rightarrow \mathcal{L}(U) = (s + \omega)E(s)$$

$$s^{-1}(U) = \mathcal{L}^{-1}(U) = e - \omega \int_0^t e \, dt$$

In integral form, $U(t) = e(t) - \omega \int_0^t e(t) \, dt$

**Example**

Plant, $G$: $\dot{x} + x = u \rightarrow G(s) = \frac{1}{s+1}$

Integral Boost: $K(s) = \frac{s + 0.1}{\omega}$ (for $\omega = 0.1$)

$|L_1| \text{ dB}$

$|L_1|$ dB is the sum of $|G_1|$ and $|K_1|$.

The integral boost increases the gain for $\omega < \omega = 0.1 \text{ rad/sec}$.
Low Frequency Boost: Use to increase gain for $\omega < \bar{\omega}$

$$K(s) = \frac{s + \bar{\omega}}{s + \frac{\bar{\omega}}{\beta}}$$ where $\bar{\omega}$ and $\beta$ are chosen constants.

Properties:
- $K(s)$ has a pole at $s = -\frac{\bar{\omega}}{\beta}$ and a zero at $s = -\bar{\omega}$
- At low frequencies, $K(\omega) \approx \frac{\bar{\omega}}{\bar{\omega}/\beta} = \beta$.
  This $\beta$ is the low frequency gain.
- At high frequencies, $K(\omega) \approx 1$.
  The low frequency boost is similar to the integral boost in that both increase the gain for $\omega < \bar{\omega}$. However, the low frequency boost "levels" off once it reaches a low frequency gain of $\beta$.

Ex) Plant $G$: $x + x = u \rightarrow G(s) = \frac{1}{s + 1}$
  Low frequency boost: $K(s) = \frac{s + 0.1}{s + 0.01}$ \((\bar{\omega} = 0.1, \beta = 10)\)
Roll off: Use to decrease gain for $\omega > \bar{\omega}$

$$K(s) = \frac{\bar{\omega}}{s + \bar{\omega}} \text{ where } \bar{\omega} \text{ is the corner frequency}$$

Properties

- $K(s)$ has a pole at $s = -\bar{\omega}$
- At low frequencies $K(s) \approx \frac{\bar{\omega}}{\bar{\omega}} = 1$
- At high frequencies $K(j\omega) \approx \frac{1}{\bar{\omega}^2}$

For every factor of 10 increase in frequency, $K(j\omega)$ decreases in gain by a factor of 10

[If $\omega_1 = 10\omega_2$ and $\omega_1, \omega_2 >> \bar{\omega}$, then $K(j\omega_1) \approx \frac{1}{10} K(j\omega_2)$]

Ex) Plant $G$: $\dot{x} + x = u \rightarrow G(s) = \frac{1}{s+1}$

Roll off: $K(s) = \frac{100}{s+100}$ ($\bar{\omega} = 100$)
Lead : Use to increase slope at $w = \bar{w}$

$$K(s) = \frac{\beta s + \bar{w}}{s + \beta \bar{w}}$$
where $\beta$ and $\bar{w}$ are chosen constants.
$\beta$ must be chosen $> 1$.

Properties
- $K(s)$ has a pole at $s = -\beta \bar{w}$ and a zero at $s = -\bar{w}/\beta$
- At low frequencies $K(j\omega) \approx \frac{\omega}{\beta \bar{w}} = \frac{1}{\beta}$
- At high frequencies $K(j\omega) \approx \beta$
- At $w = \bar{w}$, $|K(j\omega)| = 1$

Note that the lead transfer function is similar to (but not exactly equal to) a PD controller with proportional gain $\bar{w}$ and derivative gain $\beta$.

PD: $u(t) = \bar{w} e(t) + \beta e(t)$

$$\rightarrow K(s) = \frac{[\beta s + \bar{w}]}{s + \beta \bar{w}}$$

PD Transfer Function, $K_{PD}(s)$

The PD transfer function is in the numerator of the lead transfer function. The lead contains an additional pole at $s = -\beta \bar{w}$.

The additional pole in $K_{PD}$ reduces the gain at high frequencies and prevents excessive noise amplification we observed with PD controllers.
Use to decrease slope at \( \omega = \bar{\omega} \)

\[ K(s) = \frac{s + \beta \bar{\omega}}{\beta s + \bar{\omega}} \]

where \( \beta \) and \( \bar{\omega} \) are chosen constants

\( \beta \) must be chosen \( > 1 \).

**Properties**

- \( K(s) \) has a pole at \( s = -\frac{\bar{\omega}}{\beta} \) and a zero at \( s = -\beta \bar{\omega} \)
- At low frequencies \( K(j \omega) \approx \frac{\bar{\omega}}{\omega} = \beta \)
- At high frequencies \( K(j \omega) \approx \frac{1}{\beta} \)
- At \( \omega = \bar{\omega} \), \( |K(j \omega)| = 1 \)