Bode Plots: Higher Order Systems

If the system is \( n \)th order then the Bode plot can be sketched by decomposing the transfer function into simple terms.

Consider the general case where \( G(s) \) has only real poles and zeros:

\[
G(s) = K \frac{(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}
\]

The poles of \( G(s) \) are at \(-p_1, -p_2, \ldots, -p_n\) and the zeros are at \(-z_1, -z_2, \ldots, -z_m\). We’ll assume they’ve been ordered:

\[
|z_1| \leq |z_2| \leq \ldots \leq |z_m|
\]

and \( |p_1| \leq |p_2| \leq \ldots \leq |p_n| \)

**Step 1** First convert this transfer function to a standard form where all terms are of the form \( 1 + \frac{c}{s} \) for some \( c \).

\[
G(s) = K \frac{z_1(1 + \frac{s}{z_1}) \cdot z_2(1 + \frac{s}{z_2}) \cdot \ldots \cdot z_m(1 + \frac{s}{z_m})}{p_1(1 + \frac{s}{p_1}) \cdot p_2(1 + \frac{s}{p_2}) \cdot \ldots \cdot p_n(1 + \frac{s}{p_n})}
\]

\[
\Rightarrow G(s) = \left[ \frac{K z_1 z_2 \ldots z_m}{p_1 p_2 \ldots p_n} \right] \frac{(1 + \frac{s}{z_1})(1 + \frac{s}{z_2}) \ldots (1 + \frac{s}{z_m})}{(1 + \frac{s}{p_1})(1 + \frac{s}{p_2}) \ldots (1 + \frac{s}{p_n})}
\]

\[
\Rightarrow G(s) = A \frac{(1 + \frac{s}{z_1})(1 + \frac{s}{z_2}) \ldots (1 + \frac{s}{z_m})}{(1 + \frac{s}{p_1})(1 + \frac{s}{p_2}) \ldots (1 + \frac{s}{p_n})}
\]
Next sketch the magnitude plot. Note that

\[
|G(j\omega)| = |A_1| \cdot \frac{|1 + j\omega/z_1| \cdot |1 + j\omega/z_2| \ldots |1 + j\omega/z_m|}{|1 + j\omega/p_1| \cdot |1 + j\omega/p_2| \ldots |1 + j\omega/p_n|}
\]

Recall that \(\log xy = \log x + \log y\) and \(\log \frac{1}{x} = -\log x\).
Thus the magnitude of \(G(j\omega)\) in dB is given by

\[
20 \log_{10} |G(j\omega)| = 20 \log_{10} |A_1| + \left[20 \log_{10} |1 + j\omega/z_1| + \ldots + 20 \log_{10} |1 + j\omega/z_m|\right] \\
+ \left[20 \log_{10} |1 + j\omega/p_1| + \ldots + 20 \log_{10} |1 + j\omega/p_n|\right]
\]

The Bode magnitude plot of \(G(j\omega)\) is equal to the sum of simple terms. Specifically you can add the Bode plots of the constant term \(|A_1|\), the real zero terms \(|1 + j\omega/z_i|\) and subtract the Bode plots for \(|1 + j\omega/p_i|\).

We've already shown how to sketch Bode plots for terms of the form \(|1 + j\omega/c|\) where \(c\) is a constant.

For \(\omega < c\) \(20 \log_{10} |1 + j\omega/c| \approx 20 \log_{10} 1 = 0\) dB

For \(\omega > c\) \(20 \log_{10} |1 + j\omega/c| \approx 20 \log_{10} |\omega/c| \Rightarrow \text{slope is} \ + 20\text{ dB/dec}

The "corner" frequency is \(\omega = c\).
Step 3

Next sketch the Bode phase plot of $G(j\omega)$. Let $\phi_G(j\omega)$ denote the phase (or angle) of $G$.

Recall that for two complex numbers $x$ and $y$,

$$\phi(x + jy) = \phi(x) + \phi(y).$$

In other words, the phase of a product of complex numbers is the sum of the phases. Also, $\phi(1/x) = -\phi(x)$.

$$\phi(G(j\omega)) = \phi(A) + \left[ \phi \left( 1 + j\omega \phi_1 \right) + \ldots + \phi \left( 1 + j\omega \phi_m \right) \right]$$

$$+ \left[ \phi \left( \frac{1}{j\omega + p_1} \right) + \ldots + \phi \left( \frac{1}{j\omega + p_n} \right) \right]$$

We've already shown how to sketch Bode phase plots for terms of the form $\phi(1+j\omega)$.

For $\omega < \omega_c \Rightarrow \phi(1+j\omega) = 0^\circ$

For $\omega > \omega_c \Rightarrow \phi(1+j\omega) = 90^\circ$

For $\omega = \omega_c \Rightarrow \phi(1+j) = 45^\circ$

This implies that the Bode phase plot of $G(j\omega)$ is equal to the sum of the phase contributions due to the constant term, real zeros, and real poles, except for the constant $A$ is a real # and hence

$\phi A = 0^\circ$ if $A > 0$ and $\phi A = 180^\circ$ if $A < 0$. 
Ex7 \[ G(s) = \frac{10(5+10)}{(s+1)(s+100)} \]

\[ G(s) \] has a zero at \( s = -10 \) and poles at \( p = -1 \) and \( p = -100 \).

**Step 1** Put \( G(s) \) in the standard form

\[ G(s) = \frac{10 - 10(s+5/3)}{(1+s) \cdot 100 (1+s/100)} \]

\[ \Rightarrow G(s) = \frac{10 - 10(s + 5/3)}{100 (1+s/100)} \]

\[ \Rightarrow G(j\omega) = 1 + \frac{j\omega/10}{(1+j\omega) (1+j\omega/100)} \]

**Step 2** Draw the Bode Magnitude plot

\[ 20 \log_{10} |G(j\omega)| = 20 \log_{10} |1 + j\omega/10| + 20 \log_{10} |1/\omega|^{-1} + 20 \log_{10} |1 + j\omega/100|^{-1} \]
Step 3

Draw the phase plot.

\[ \frac{\Delta G(\omega)}{\Delta G(\omega)} = 4(1 + \frac{1}{\omega/\omega_0}) + 4(1 + \omega) + 4(1 + \frac{\omega/\omega_0}{\omega/\omega_0})^{-1} \]

4G in deg

4G is the sum of the 3 phases.

This plot is only a straight line approximation of the true Bode plot.
A similar procedure can be used to sketch the Bode plot for systems that are higher order and have complex poles/zeros.

\[ G(s) = \frac{s + 0.01}{s^2 + 0.15s + 1} = \frac{0.01}{s + 0.101} \]

This transfer function has a zero at \( z = -0.01 \) and a lightly damped complex pair of poles with \( \omega_n = 1 \) and \( 2\pi\omega_n = 0.1 \rightarrow f = 0.05 \). Note!

\[
20 \log_{10} |G(j\omega)| = 20 \log_{10} 0.01 + 20 \log_{10} \left| \frac{1}{1 + \omega^2/0.101^2 + 1} \right|
\]

\[-40 \text{ dB} \text{ because} \]

\[ \log_{10} 0.01 = -2 \]

The Bode magnitude plot is thus the sum of 3 terms:

\[
\text{Magnitude (dB)} \quad 11 + \omega/0.011 \text{ in dB}
\]

Slope is \( \omega \text{dB/dec} \)

A more accurate sketch of the second order term includes the resonant peak \( \frac{1}{2\pi} = 20 \text{ dB/dec} \text{ (Error)} \)

The approx peak of \( \frac{1}{2\pi} \) is accurate if \( f \) is small \( (f < 1) \)

Slope is \( \omega \text{dB/dec} \)

\[ -40 \text{ dB} \text{/dec} \]

\[ -40 \text{ dB} \]

\[ -10 \text{ dB} \text{/dec} \]

\[ 0 \text{ dB} \]

\[ 20 \text{ dB} \]

\[ -80 \text{ dB} \]
Add these 3 Bode Magnitude plots to obtain the final sketch of $G$

$|G| \text{ in dB}$

Include resonant peak for more accurate sketch.

Slope is $-20 \text{ dB/dec}$

To draw the phase plot of $G$ note:

$\phi G = \phi 0.01 + \phi (1 + j\omega / 0.01) + \phi \left( \frac{1}{(j\omega)^2 + 0.1j\omega + 1} \right)$

Thus the phase plot is the sum of 2 terms:

$\phi G (\text{deg})$

A more accurate sketch:

If $\omega$ is then the phase transition is shift.
Sum the phase plots for the two terms to get the phase plot for \( \angle G \)