We have seen that the steady-state frequency response of a system is governed by the system transfer function \( G(s) \) evaluated at \( s = jw \) (where \( w \) is the freq. of the input sinusoid).

It is useful to have graphical displays of \( G(jw) \) in order to understand how a system responds to sinusoids of different frequencies.

Two common frequency response plots

A) Bode Plot: \( G(jw) \) is a complex number with magnitude \( |G(jw)| \) and angle (phase) \( \angle G(jw) \). The magnitude and angle depend on the frequency \( w \). A Bode plot consists of two plots

- \( |G(jw)| \) vs. \( w \):
- \( \angle G(jw) \) vs. \( w \)

It can be generated in Matlab with the BODE command. The plot is named after Hendrik Bode.

B) Nyquist Plot: \( G(jw) \) is a complex number with real part \( \text{Real}[G(jw)] \) and imaginary part \( \text{Imag}[G(jw)] \). Again, the real/imaginary parts depend on frequency \( w \). A Nyquist plot is a single plot of \( \text{Real}[G(jw)] \) vs. \( \text{Imag}[G(jw)] \).

It can be generated in Matlab with the NYQUIST command. The plot is named after Harry Nyquist.

We'll start with the Bode plot and come back to the Nyquist plot later.
The two subplots of a Bode plot are typically drawn as:

a) Magnitude plot: - Horizontal axis: \( \omega \) on a log-scale 
   - Vertical axis: \( 20 \log_{10} |G(\omega)| \)

b) Phase Plot: - Horizontal axis: \( \omega \) on a log-scale 
   - Vertical axis: \( \angle G(\omega) \) in degs (or in radians)

The quantity \( 20 \log_{10} |G(\omega)| \) is the Magnitude of \( |G(\omega)| \) in units of decibels. Some useful conversions are:

- 20 dB = Gain of 10 \( (20 \log_{10} 10 = 20) \)
- 6 dB = Gain of 2 \( (20 \log_{10} 2 \approx 6) \)
- 0 dB = Gain of 1 \( (20 \log_{10} 1 = 0) \)
- -6 dB = Gain of \( \frac{1}{2} \) \( (20 \log_{10} \frac{1}{2} \approx -6) \)
- -20 dB = Gain of \( \frac{1}{10} \) \( (20 \log_{10} 0.1 = -20) \)

We'll spend some time reviewing how to draw Bode plots. You can always generate the Bode plot of a complicated system using the BODE MATLAB command. However, it is important to have a good feel for how the Bode plot is related to the system ODE/transfer function. In addition, you must build some intuition about how the plot is related to the transient response characteristics of the system. This intuition will be used to design and analyze control systems.
First-Order Systems

\[ \dot{x} + ax = bu \]

Recall a few facts from our previous analysis of first-order systems:

- The system is stable if \( a > 0 \) [Note that we have written the "ax" term on the left side of the ODE].

- If \( u(t) = \bar{u} \) for \( t > 0 \) (where \( \bar{u} \) is some constant)
  then \( x(t) \to x_{ss} = \left( \frac{b}{a} \right) \bar{u} \) as \( t \to \infty \)
  (Assuming the system is stable)

- The time constant for this system is \( T = \frac{1}{a} \).
  (Again, we are assuming the system is stable so \( a > 0 \)).
  Thus, moreover, the settling time is \( t_s = 3T = \frac{3}{a} \).
  Thus the speed of response \( T \) is \( \frac{1}{a} \),
  i.e., larger values of \( a \) correspond to faster response.

We construct the transfer function for the system
by taking the Laplace transform of the ODE assuming
zero initial conditions.

\[ s \dot{x}(s) + ax(s) = bu(s) \]
\[ \Rightarrow x(s) = \frac{b}{s+a} u(s) \]

\[ \text{TF} \quad G(s) = \frac{x(s)}{u(s)} = \frac{b}{s+a} \]

Notice that \( G(s) \) has a pole at \( s = -a \).
The system is stable if this pole is in the LHP, i.e., \( a > 0 \).
To construct the Bode plot we need to evaluate the magnitude and angle of \( G(s) \) at \( s = j\omega \).

\[
G(j\omega) = \frac{b}{j\omega + a} = \frac{b}{a + j\omega} \cdot \frac{a - j\omega}{a - j\omega} = \frac{ab - j\omega b}{a^2 + \omega^2}
\]

(Clear the complex # from the denominator by multiplying the top/bottom by the conjugate of the bottom)

This complex # can be visualized in the complex plane

\[
\begin{align*}
|G(j\omega)| &= \sqrt{\left(\frac{ab}{a^2 + \omega^2}\right)^2 + \left(\frac{-\omega b}{a^2 + \omega^2}\right)^2} = \sqrt{\frac{b^2}{a^2 + \omega^2}} \\
\Rightarrow |G(j\omega)| &= \frac{b}{\sqrt{a^2 + \omega^2}} \\
\Rightarrow 20 \log_{10} |G(j\omega)| &= 20 \log_{10} \frac{b}{\sqrt{a^2 + \omega^2}}
\end{align*}
\]

\[
\phi G(j\omega) = \tan^{-1}\left[\frac{-\omega b / a}{ab / (a^2 + \omega^2)}\right] = -\tan^{-1}\left[\frac{\omega}{a}\right]
\]
The dependence of $|G(jw)|$ and $\angle G(jw)$ on $w$ is easier to understand by considering the asymptotes:

- $w < a$:
  \[
  G(jw) = \frac{b}{jw + a} \approx \frac{b}{a}
  \]
  \[
  |G(jw)| \approx \frac{b}{a} \quad \text{and} \quad \angle G(jw) \approx 0 \text{ for } w < a. \quad (\text{if } b > 0)
  \]

- $w > a$:
  \[
  G(jw) = \frac{b}{jw + a} \approx \frac{b}{jw} = -j\frac{b}{w}
  \]
  \[
  |G(jw)| \approx \frac{b}{w} \quad \text{and} \quad \angle G(jw) \approx 90^\circ \quad (\text{if } b > 0)
  \]
  \[
  \text{Note: } 20\log_{10}|G(jw)| = 20\log_{10}b - 20\log_{10}w
  \]

- $w = a$:
  \[
  G(ja) = \frac{b}{ja + a} = \frac{b}{a} \cdot \left(\frac{1}{1+j}\right)
  \]
  \[
  |G(ja)| = \frac{b}{\sqrt{2}a} \quad \text{and} \quad \angle G(ja) = -45^\circ \quad (\text{if } b > 0)
  \]

---

**Gain at $w=0$ is $G(0) = \frac{b}{a}$. This is the DC gain.**

**Note how this relates to the steady-state response to a step input.**

---

**The high frequency asymptote has a slope of $-20\,\text{dB/decade}$ because $20\log_{10}\frac{1}{w} = -20\log_{10}w$.**

This is the "corner" frequency (also the "breakdown")

**Note that $a$ corresponds to a faster system; a higher corner frequency.**

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**Straight-line approximation based on asymptotes.**

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**Actual plot**
Ex) \[ x + 10x = 10u \Rightarrow G(s) = \frac{10}{s + 10} \]

\[ |G| \text{ (dB)} \]

- Note: \( G(s) = 1 \) so \( 20 \log_{10} |G| = 0 \text{ dB} \)
- Also: The corner freq is at \( \omega = 10 \text{ rad/sec} \)

\[ \frac{1}{1 + j} \Rightarrow \left\{ \begin{array}{l} |G(j\omega)| = \frac{1}{\sqrt{2}} \Rightarrow 20 \log_{10} |G(j\omega)| = -3 \text{ dB} \vspace{10pt} \\ \frac{\pi}{4} G(j\omega) = -90^\circ \end{array} \right. \]

Ex) \[ x - 10x = 10u \Rightarrow G(s) = \frac{10}{s - 10} \]

This system is unstable. The RHP pole does not affect the magnitude plot because \( |\omega - 10| = |\omega + 10| \). However, it does affect the phase. Consider the asymptotes:
- \( \omega < 0 \Rightarrow G_u(j\omega) \approx \frac{10}{j\omega} \rightarrow \left\{ \begin{array}{l} |G_u(j\omega)| \approx 1 \vspace{10pt} \\ \theta G_u(j\omega) \approx -180^\circ \end{array} \right. \)
- \( \omega > 10 \Rightarrow G_u(j\omega) \approx \frac{10}{j\omega} \rightarrow \left\{ \begin{array}{l} |G_u(j\omega)| \approx \frac{10}{\omega} \vspace{10pt} \\ \frac{\pi}{4} G_u(j\omega) \approx -90^\circ \end{array} \right. \)
Ex) Consider the PD Control

\[ u(t) = 100e + 1000\dot{e} \]

\( u(t) \) \quad \text{Derivative Gain}

\( \dot{e} \) \quad \text{Proportional Gain}

We construct the transfer function by taking the Laplace Transform assuming zero initial condition.

\[ u(s) = 100E(s) + 100sE(s) \]

\[ \Rightarrow u(s) = (100 + 100s)E(s) \]

\[ G(s) = \frac{u(s)}{E(s)} = 100s + 10 = 100(s + \frac{1}{10}) \]

Notice that \( G(s) \) has a zero at \( s = -\frac{1}{10} \) and it has no poles (it is "non-proper"). We can sketch the Bode plot by considering the low and high frequency asymptotes:

- \( W << \frac{1}{\zeta} \Rightarrow G(j\omega) = 100(j\omega + \frac{1}{10}) \approx 100(j\omega) = 10 \)

\[ |G(j\omega)| \approx 10 \text{ and } \angle G(j\omega) \approx 0 \text{ for } \omega << \frac{1}{\zeta} \]

- \( W >> \frac{1}{\zeta} \Rightarrow G(j\omega) = 100(j\omega + \frac{1}{10}) \approx 100j\omega = j(100\omega) \)

\[ |G(j\omega)| \approx 100\omega \text{ and } \angle G(j\omega) \approx +90^\circ \text{ for } \omega >> \frac{1}{\zeta} \]

[Note: \( 20\log_{10} |G(j\omega)| = 20\log_{10}(100) + 20\log_{10}\omega = 40 + 20\log_{10}\omega \)]
First-order System with Real Zero

We can draw easily draw sketches of Bode Plots for more complicated systems by using the following fact of complex #s:

Let \( W_1 = r_1 e^{j\phi_1} \) and \( W_2 = r_2 e^{j\phi_2} \) be 2 complex numbers in polar form. Then:

\[
W_1 W_2 = r_1 r_2 e^{j(\phi_1 + \phi_2)}
\]

Multiplying 2 complex #s gives another complex # w/ the product of the magnitudes and the sum of the phases.

\[\text{Ex}\]

\[0.1 \dot{u} + 1000 u = e + 1000 e \quad \text{\(\rightarrow\)} \quad \text{Lead Control} \quad u\]

This particular system is called a "lead" control law and it appears on your homework.

Take the Laplace transform to find the transfer function:

\[
0.15 (U(s) + 1000 u) = 5 E(s) + 1000 E(s)
\]

\[
[0.15 + 1000] U(s) = [5 + 1000] E(s)
\]

\[
K(s) = \frac{U(s)}{E(s)} = \frac{5 + 1000}{0.15 + 1000}
\]

This control law has a zero at \( z = -1000 \)

and a pole at \( p = -10,000 \).
To sketch the Bode plot, first notice that

\[ 20 \log |K(j\omega)| = 20 \log |(j\omega + 1000)| + 20 \log \left| \frac{1}{0.1j\omega + 1000} \right| \]

\[ 4|K(j\omega)| = 4(j\omega + 1000) + 4(0.1j\omega + 1000) \]

In other words, we can sketch the Bode magnitude and phase for each individual term (the zero and pole) and then simply sum the results. This strategy works even if there are no poles and no zeros. See Ogata for the more general case.

**Note:**

\[ 4K(j\omega) > 0 \text{ for all } \omega \]

so if the input is \( e(t) = A \sin(\omega t) \),

then the output is \( u(t) = A|K(j\omega)| \sin(\omega t + \angle K(j\omega)) \).

Since \( 4K(j\omega) > 0 \),

the output sinusoid will be ahead (it will lead) of the input.