Consider a signal \( u(t) \) defined on \( t \geq 0 \).

The one-sided Laplace transform of \( u(t) \) is defined by
\[
U(s) = \int_{0}^{\infty} u(t) e^{-st} \, dt
\]
where \( s \) is a complex number (denoted \( s \in \mathbb{C} \)). As notation, we'll write \( U(s) = \mathcal{L}\{u(t)\} \).

The Laplace transform is defined at values of \( s \) for which the integral converges. This is known as the region of convergence.

A value \( s = \alpha \) such that \( U(s) = \int_{0}^{\infty} e^{-s t} \, dt \) is called a pole of \( U(s) \).

Ex) \( u(t) = e^{2t} \) for \( t \geq 0 \)

\[
U(s) = \int_{0}^{\infty} e^{2t} e^{-st} \, dt = \int_{0}^{\infty} e^{2t} e^{-(s-2)t} \, dt = \int_{0}^{\infty} e^{(2-s)t} \, dt
\]

\[
= \frac{1}{2-s} e^{(2-s)t} \bigg|_{t=0}^{t=\infty}
\]

\[
= \lim_{t \to \infty} \left( \frac{1}{2-s} e^{(2-s)t} \right) - \left( \frac{1}{2-s} \right)
\]

\[
e^{(2-s)t} = e^{Re(2-s)t} \quad e^{lm(2-s)t} \to 0 \quad \text{if} \quad Re(2-s) \leq 0 \quad (e \to Re(s) \geq 2)
\]

\[
\Rightarrow \quad U(s) = \frac{1}{s-2} \quad \text{for} \quad s \text{ that satisfy} \quad Re(s) > 2,
\]

\[
\text{5 is complex and} \quad U(s) \text{ is complex. This makes it difficult to visualize this function. Let's simply plot the magnitude of} \quad U(s) \text{ for} \quad |s| \\
\text{for} \quad s=2. \text{ } \quad \Rightarrow \quad \text{pole} \text{ because the plot looks like a tent held up by a pole at} \ s=2.
\]
Similarly, you can show from the definition that the signals in the table below have the given Laplace transforms.

<table>
<thead>
<tr>
<th>Function</th>
<th>Time Domain</th>
<th>Laplace Domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unit Impulse</td>
<td>$u(t) = 1$</td>
<td>$\frac{1}{s}$</td>
</tr>
<tr>
<td>Unit Step</td>
<td>$u(t) = 1$</td>
<td>$\frac{1}{s}$</td>
</tr>
<tr>
<td>Unit Ramp</td>
<td>$u(t) = t$</td>
<td>$\frac{1}{s^2}$</td>
</tr>
<tr>
<td>$N^{th}$ Power of $t$</td>
<td>$u(t) = t^n$</td>
<td>$\frac{n!}{s^{n+1}}$</td>
</tr>
<tr>
<td>Exponential</td>
<td>$u(t) = e^{-at}$</td>
<td>$\frac{1}{s+a}$</td>
</tr>
<tr>
<td>Sine</td>
<td>$u(t) = \sin wt$</td>
<td>$\frac{w}{s^2+w^2}$</td>
</tr>
<tr>
<td>Cosine</td>
<td>$u(t) = \cos wt$</td>
<td>$\frac{s}{s^2+w^2}$</td>
</tr>
<tr>
<td>First order step response</td>
<td>$u(t) = 1 - e^{-at}$</td>
<td>$\frac{a}{s(s+a)}$</td>
</tr>
<tr>
<td>Exponentially Decaying Sine</td>
<td>$u(t) = e^{-at} \sin wt$</td>
<td>$\frac{w}{(s+a)^2 + w^2}$</td>
</tr>
<tr>
<td>Exponentially Decaying Cosine</td>
<td>$u(t) = e^{-at} \cos wt$</td>
<td>$\frac{st+a}{(s+a)^2 + w^2}$</td>
</tr>
</tbody>
</table>
The inverse Laplace transform of \( u(s) \) is defined as

\[
u(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} u(s) e^{st} \, ds
\]

The notation means we integrate along a line in the complex plane with \( \text{Re}(s) = c \) and \( \text{Im}(s) = \infty \), \( c \) must be chosen large enough that \( \text{Re}(s) > c \) is in the region of convergence of \( u \).

As notation: \( u(t) = \mathcal{L}^{-1}\{u(s)\} \).

We won't need to apply the formula for the inverse Laplace transform. Instead, we'll rely on tables of Laplace transforms to convert between the time and Laplace domains.

However, you should be aware that we are sweeping some mathematical technicalities under the rug.

**Basic Properties of the Laplace Transform**

There are many interesting properties of the Laplace transform. An important property for our purposes involves the Laplace transform of \( \frac{dx(t)}{dt} \):

\[
\mathcal{L}\left\{ \frac{dx(t)}{dt} \right\} = s X(s) - x(0)
\]

Proof:

\[
\mathcal{L}\left\{ \frac{dx(t)}{dt} \right\} = \int_{0}^{\infty} \frac{dx(t)}{dt} e^{-st} \, dt = e^{-st} x(t) \bigg|_{0}^{\infty} - \int_{0}^{\infty} e^{-st} \, dx(t) = [-x(t)]_{0}^{\infty} + s \int_{0}^{\infty} x(t) e^{-st} \, dt
\]

\[
= [-x(0)] + s \int_{0}^{\infty} x(t) e^{-st} \, dt = s X(s) - x(0)
\]

Use integration by parts:

\[
\text{Sdv = uv - vdu}
\]

with \( u = e^{-st}, dv = x(t) \, dt \)

\[
du = -se^{-st} \, dt, \quad v = x(t)
\]
Some other important properties of the Laplace transform:

**Linearity**
If \( a \) and \( b \) are numbers and \( x_1(t) \) and \( x_2(t) \) are time domain signals, then
\[
\mathcal{L}\{ax_1 + bx_2\} = a\mathcal{L}\{x_1\} + b\mathcal{L}\{x_2\}
\]

**Example**
\[
\mathcal{L}\{e^{2t}\} = \frac{1}{s-2}
\]
\[
\mathcal{L}\{\sin 5t\} = \frac{5}{s^2 + 25}
\]
\[
\therefore \mathcal{L}\{9e^{2t} + 17\sin 5t\} = \frac{9}{s-2} + \frac{17.5}{s^2 + 25}
\]

**Differentiation** We just showed that
\[
\mathcal{L}\{x(t)\} = sX(s) - x(0)
\]
More generally:
\[
\mathcal{L}\{x'(t)\} = sX(s) - x(0) - x'(0)
\]
\[
\mathcal{L}\{x''(t)\} = s^2X(s) - sx(0) - x'(0)
\]
\[
\mathcal{L}\{x^{(n)}(t)\} = s^nX(s) - s^{n-1}x(0) - \ldots - \frac{d^{n-1}x}{dt^{n-1}}(0)
\]

**Example** Let \( x(t) = \cos 6t \). Then applying this rule
\[
\mathcal{L}\{x(t)\} = s\mathcal{L}\{\cos 6t\} - x(0) = s\left[\frac{6}{s^2 + 36}\right] - 1
\]
\[
= \frac{6}{s^2 + 36} - \frac{6}{s^2 + 36}
\]
\[
= \frac{6}{s^2 + 36}
\]

Notice that \( \mathcal{L}\{x(t)\} = -6 \left[ \frac{6}{s^2 + 36} \right] \) on multiplying by \( s \).

Thus \( \mathcal{L}\{x'(t)\} = \mathcal{L}\{-6\sin 6t\} \)

This is expected since \( x(t) = \cos 6t \Rightarrow x' = -6\sin 6t \).
The linearity and differentiation properties of the Laplace transform can be used to solve ODEs.

\[ \dot{X}(t) = -2X + 6U \]
\[ X(0) = 4 \]
\[ u(t) = \begin{cases} 
0 & t < 0 \\
3 & t \geq 0 
\end{cases} \]

Based on our previous work on first-order systems, we know:

a) The system is stable with time constant \( T = \frac{1}{2} \).

Thus the response should settle to the final value in \( 3T = 1.5 \) sec.

b) The steady-state value of \( X \) satisfies

\[ 0 = -2x_{ss} + 6u \Rightarrow x_{ss} = 3u = 9 \]

Thus we expect the solution to look like

To solve this with Laplace transform, apply the linearity and differentiation property to get:

\[ \mathcal{L} \{ \dot{X} \} = \mathcal{L} \{ -2X + 6U \} \]

\[ \Rightarrow \]

\[ sX(s) - x(0) = -2X(s) + 6U(s) \]

\[ \Rightarrow \]

\[ (s+2)X(s) = x(0) + 6U(s) \]

\[ \Rightarrow \]

\[ X(s) = \frac{x(0)}{s+2} + \frac{6}{s+2}U(s) \]
From the Laplace transform tables,
\[ U(s) = \mathcal{L} \{ u(t) \} = \frac{3}{s} \]
Moreover, \( x(0) = 4 \). Thus
\[ X(s) = \frac{4}{s+2} + \frac{18}{s(s+2)} \]

Now take the inverse Laplace transform of each side and use the Laplace transform tables to get \( x(t) \):
\[ \begin{align*}
    x(t) &= \mathcal{L}^{-1} \{ X(s) \} \\
     &= \mathcal{L}^{-1} \left\{ \frac{4}{s+2} + \frac{18}{s(s+2)} \right\} \\
     &= 4 \mathcal{L}^{-1} \left\{ \frac{1}{s+2} \right\} + 9 \mathcal{L}^{-1} \left\{ \frac{2}{s(s+2)} \right\} \\
     &= 4e^{-2t} + 9(1-e^{-2t}) \\
\Rightarrow x(t) &= 9 - 5e^{-2t}
\end{align*} \]

Note that \( x(0) = 4 \); hence, \( x(t) \rightarrow 9 \) as \( t \rightarrow \infty \)
and the exponential term is \( e^{-2t} \approx e^{-3} \approx 0.05 \) when \( t = 1.5 \text{ sec} \).
This agrees with our expected solution.