Consider the first-order system:

\[
(x1) \quad \dot{x} + ax = bu
\]

We previously looked at a proportional controller:

\[
(x2) \quad u(t) = K_p \left( \frac{r(t) - x(t)}{\text{error, etc}} \right)
\]

The closed-loop dynamics from reference input \( r \) to system response \( x \) is given by plugging \( u \) from (x2) into (x1):

\[
\dot{x} + ax = bK_p [r - x]
\]

\[
(x3) \quad \dot{x} + (a + bK_p)x = bK_p r
\]

The closed-loop dynamics (x3) are stable if \( a + bK_p > 0 \) and the time constant is \( T = \frac{1}{a + bK_p} \). If the closed-loop is stable and \( r(t) = \bar{r} \) for \( t \geq 0 \), then \( x \) will converge to a steady-state value \( x_{ss} \):

\[
\dot{x} + (a + bK_p)x_{ss} = bK_p \bar{r}
\]

\[\Rightarrow x_{ss} = \frac{bK_p}{a+bK_p} \bar{r}\]

\( x \) does not converge to \( \bar{r} \), i.e., there will be steady-state error. The steady-state error is given by:

\[
e_{ss} = \bar{r} - x_{ss} = \left[ 1 - \frac{bK_p}{a+bK_p} \right] \bar{r}
\]

\[\Rightarrow e_{ss} = \frac{a}{a+bK_p} \bar{r}\]

To recap, the steady-state error decreases as \( K_p \) increases. We can make \( |e_{ss}| \) as small as we like by choosing \( K_p \) sufficiently large.
One issue with large proportional gains is that they lead to large values of the input, \( u \). Typically there are practical limits on how large we can make \( u \) without damaging the system.

To understand this issue, consider the closed-loop response due to a step reference command \( \mathbf{r}(t) = \begin{cases} 0 & t < 0 \\ \bar{r} & t \geq 0 \end{cases} \)

Since \( e = r - x \), its response

starts at \( e(0) = \bar{r} - x(0) = \bar{r} \)

and converges to \( e_{ss} \)

Since \( u = K_p e \)

\( u(t) = K_p e(t) = K_p \bar{r} \)

Note that even though \( e \to e_{ss} \neq 0 \), \( u_{ss} \) remains at \( \bar{r} \). \( u \) only depends on the current value of \( e \). It does not change even if the system has been at \( e \) for a long period of time.

The maximum value of \( |u| \) is \( |K_p \bar{r}| \)

and this occurs at \( t = 0 \). \( K_p \) will also \( \uparrow \) the maximum value of \( u \).
If there is a requirement that $|u| \leq \text{max}$, then this will limit how large we can choose $K_p$.
Consequently, it will limit how small we can make $\text{ess}$. Note that the largest value of $|u(t)|$ on $p^2$ occurred at $t=0$. However, the steady-state error is governed by the value of $u$ as $t \to \infty$. It seems like we should be able to make $\text{ess}$ small and still keep $|u|$ from getting too large.

A **proportional-integral (PI)** control law can be used to eliminate steady errors that occur due to constant step changes in reference.

$$u(t) = K_p (\tau(t) - x(t)) + K_i \int_0^t [(r(\tau) - x(\tau))] d\tau$$

This term depends on the integral of the error.

As a thought experiment, consider the effect of the integral term on proportional system response on $p^2$.

The integral is the area under this curve.

As time goes on, the area increases and hence $u$ will increase. This will eventually force $e \to 0$. 

\[ u(t) = K_p (r(t) - x(t)) + K_i \int_0^t (r(\tau) - x(\tau)) \, d\tau \]

The integral term reacts to the past as measured by the accumulated integral of the tracking error.

The proportional term reacts to the present as measured by the current tracking error.

We can use our knowledge of second order systems to get a better understanding of how the gains \( K_p \) and \( K_i \) affect the system response.

**Plant:**

\[ \dot{x} + ax = bu \]

**PI Control:**

\[ u(t) = K_p (r(t) - x(t)) + K_i \int_0^t (r(\tau) - x(\tau)) \, d\tau \]

A block diagram of the closed-loop system is:

Notice that the plant and controller are both dynamical systems. As the individual systems become more complicated it gets useful to group them into subsystems. You can do this in Simulink with the "Subsystem" block in the "Ports and Subsystems" folder. This is similar to using functions in programming to modularize code.
The closed-loop system has one external input \( r \) and one output \( x \). We can combine the models for the system (plant) and controller to obtain a single ODE that describes the relationship between \( r \) and \( x \).

Plug the PI controller into the plant ODE:

\[
\dot{x} + ax + b \left[ k_p (r - x) + k_i \int_0^t (r - x) \, dt \right]
\]

Differentiate both sides with respect to \( t \) to get rid of the integral:

\[
\ddot{x} + a \dot{x} + b k_p [r - x] + k_i [r - x]
\]

Next, collect all the terms that depend on \( x \):

\[
\ddot{x} + [a + bk_p] \dot{x} + k_i x = bk_p \dot{r} + k_i r
\]

Notice that the forcing term on the right side depends on both \( r \) and \( \dot{r} \).

To simplify the analysis, we'll consider the case where \( r \) is a constant \( r(t) = \bar{r} \). In this case, the term involving \( \dot{r} \) drops out (it is zero). We'll revisit the impact of the \( \dot{r} \) term later in the course.
Assuming $\dot{r} = 0$, the ODE for the closed-loop system is:

$$
\dot{X} + \left[ -a + bK_p \right] X + K_i X = K_i \dot{r}
$$

The system is stable (free response decays to zero if:

$$
\begin{align*}
& a + bK_p > 0 \\
& K_i > 0
\end{align*}
$$

For now, we'll assume $b > 0$. In this case, the system is stable if the gains satisfy:

$$
K_p > \frac{a}{b} \\
K_i > 0
$$

Next, note that if the system is stable and $r(t) = \bar{r}$
then $X(t)$ will converge to a steady-state value $X_{ss}$. In steady-state, both $\dot{X}$ and $\dot{r}$ are zero. Thus $X_{ss}$ must satisfy:

$$
K_i \dot{X}_{ss} = K_i \bar{r} \Rightarrow X_{ss} = \bar{r} \Rightarrow e_{ss} = X_{ss} - \bar{r} = 0
$$

$X(t)$ converges to $\bar{r}$ and there is no steady-state error.

This is a key property of integral control that holds for any steady-state.

Integral: If the system converges to a steady-state, then $e_{ss} = 0$.

**Proof**

Note that the RC controller is:

$$
\dot{u}(t) = K_p e(t) + K_i \int_{0}^{t} e(\zeta) d\zeta
$$

Assume there is a time $t_0$ such that $\int_{0}^{t_0} u(\zeta) d\zeta = u_{ss}$ for $t > t_0$.

Then $(\ast)$ implies for any $t > t_0$:

$$
\begin{align*}
\dot{u}_{ss} &= u(t) \\
&= K_p e_{ss} + K_i \int_{0}^{t} e(\zeta) d\zeta \\
&= K_p e_{ss} + K_i \int_{0}^{t_0} e(\zeta) d\zeta + K_i \int_{t_0}^{t} e(\zeta) d\zeta \\
&= u(0) + K_i e_{ss} (t - t_0) \\
&= u_{ss} + K_i e_{ss} (t - t_0)
\end{align*}
$$

$$
\Rightarrow K_i e_{ss} (t - t_0) = 0 \Rightarrow e_{ss} = 0
$$
If the closed loop is stable then
\[ \dot{x} + \left[ \frac{a + bK_p}{2\omega_n} \right] \dot{x} + K_i x = K_i r \]
\[ \omega_n^2 = K_i \quad \Rightarrow \quad \omega_n = \sqrt{K_i} \]
\[ 2\pi \omega_n = [a + bK_p] \quad \Rightarrow \quad \frac{1}{\omega_n} = \frac{1}{2\pi} \left[ \frac{a + bK_p}{2\sqrt{K_i}} \right] \]
\[ \Rightarrow \quad \tau = \frac{[a + bK_p]}{2\sqrt{K_i}} \]

Next we can determine how the gains \( K_p \) and \( K_i \) affect the unit step response of the closed-loop system.

We can solve for the damping ratio and natural frequency:

If we continue to assume \( b > 0 \) then we notice:

1) \( \uparrow K_p \Rightarrow \uparrow \tau, \text{ and } \uparrow \frac{1}{\omega_n} \)

2) \( \uparrow K_i \Rightarrow \uparrow \frac{1}{\omega_n} \text{ and } \uparrow \tau \text{ but no effect on } \frac{1}{\omega_n} \)

We can use the relations to understand the impact of \( K_p/K_i \) on overshot and settling time. Recall that the second-order response has 3 cases depending on \( \tau \). Let’s start by considering the underdamped case \((\tau < 1)\). Recall that for an underdamped system

\[ M_p = \text{peak overshoot} = \frac{y(t)_{\text{peak}} - y_{\text{ss}}}{y_{\text{ss}}} = e^{-\frac{\tau}{\sqrt{1-\tau^2}} \pi} \]

\[ \tau_{s} = 5\% \text{ settling time} \approx \frac{3}{\omega_n} \]

Thus \( K_p \) and \( K_i \) have the following effects:

1) \( \uparrow K_p \Rightarrow \uparrow \tau \text{ and } \uparrow \frac{1}{\omega_n} \Rightarrow \uparrow M_p \text{ and } \uparrow \tau_s \)

2) \( \uparrow K_i \Rightarrow \uparrow \tau \text{ and no effect on } \frac{1}{\omega_n} \Rightarrow \uparrow M_p \text{ and no effect on } \tau_s \).
If the system is overdamped (\( \tau > 1 \)) then there is no overshoot. The roots of the characteristic equation are at:

\[
\begin{align*}
\Gamma_1 &= -\mu_n + \omega_n \sqrt{\tau^2 - 1} \\
\Gamma_2 &= -\mu_n - \omega_n \sqrt{\tau^2 - 1}
\end{align*}
\]

As \( \tau \uparrow \) the slower root at \( \Gamma_1 \) converges to zero (it gets slower). Thus for an overdamped system, \( K_p \) and \( K_i \) have the effects:

1) \( \uparrow K_p \Rightarrow \uparrow \tau_s \Rightarrow \uparrow t_s \) (system becomes slower)

2) \( \uparrow K_i \Rightarrow \downarrow \tau \Rightarrow \downarrow t_s \) (system becomes faster)